UNCLASSIFIED

VOLUME IX, NUMBER 1 FEBRUARY, 1956

Communications on PURE AND APPLIED MATHEMATICS

A JOURNAL ISSUED QUARTERLY BY
THE INSTITUTE OF MATHEMATICAL SCIENCES
NEW YORK UNIVERSITY

INTERSCIENCE PUBLISHERS, INC., NEW YORK

Communications on

PURE AND APPLIED MATHEMATICS

Issued by the Institute of Mathematical Sciences

Natascha Artin, Technical Editor — Richard Courant, Director

Volume IX, Number 1 **CONTENTS** **February, 1956**

Many of these papers are technical reports of the research organizations of the United States Army, Navy, and Air Force.

Issued quarterly — Subscription price $8.00 per year — Foreign postage $0.50

Subscriptions should be sent to

INTERSCIENCE PUBLISHERS, INC., 250 Fifth Ave., New York 1, N.Y.

or to

INTERSCIENCE PUBLISHERS LTD., 88/90 Chancery Lane, London, W.C. 2

PRINTED IN THE NETHERLANDS BY KONINKLIJKE VERENIGDE DRUKKERIJEN HOITSEMA N.V. GRONINGEN

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. IX, 1–31 (1956)

Free Boundary Problem for the Heat Equation with Applications to Problems of Change of Phase*

I. GENERAL METHOD OF SOLUTION

I. I. KOLODNER

In the physics of change of phase processes (melting, evaporation, recrystallization, dissolution) one often encounters a boundary value problem for the heat equation in a domain whose boundary is not fully known but which must be found. It is customary to call such boundaries free boundaries.

1. The Problem

Let D_R be an open and—for simplicity of the argument—a simply connected domain in the x, t-plane. Let its boundary, as in Figure 1, consist of: the unknown arc AB represented by the equation $x = R(t)$, the given arc AC, the segment BC of the characteristic $t = \bar{t}$.

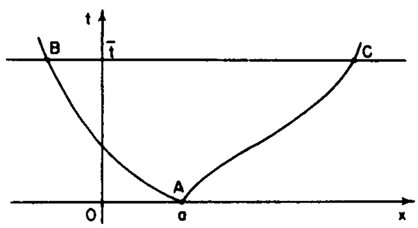

Figure 1

The coordinates were so chosen that the unknown arc meets the given arc at a point $(a, 0)$. Without loss of generality we may assume that $a = 0$, although in some cases—as in examples 3–5 of Section 6—it is convenient to have $a \neq 0$.

The problem is to find $R(t)$, $0 \leqq t \leqq \bar{t}$, and $u(x, t) = u(P)$, $P \in D$, satisfying[1]:

---

*This paper represents results obtained under Contract N6ori-201, T. O. No. 1, sponsored by the Office of Naval Research.

[1]The following abbreviations are used throughout: DE—differential equation, IC—initial condition, BC—boundary condition, FBC—free boundary condition, JC—jump condition, ∞C—condition for large $|x|$.

$$
\begin{array}{llll}
 & \text{DE} & Lu \equiv u_{xx} - u_t = 0, & P \in D_R, \\
 & \text{BC} & \Gamma u = 0 & \text{on arc } AC, \\
(1.1) & \text{FBC1} & u(R, t) = f(t) & \text{for } 0 \leqq t \leqq \bar{t}, \\
 & \text{FBC2} & u_x(R, t) = g(t) & \text{for } 0 < t \leqq \bar{t}.
\end{array}
$$

As to the functions $f(t)$ and $g(t)$, we shall not require that they be explicitely given as functions of t, but permit them to be functionals of $R(t)$. In all the examples considered in Section 6, they are functions of $R(t)$, $\dot{R}(t)$ etc. We shall occasionally use the notation $f[R]$, $g[R]$. We assume that if $\varrho(t)$ is continuously differentiable in the half-open interval $(0, \bar{t}]$ and

$$(1.2) \qquad |\dot{\varrho}| \leqq At^{-1/2}, \qquad \varrho(0) = a,$$

then $f[\varrho]$ is, as *a function of t*, continuously differentiable in the half-open interval $(0, \bar{t}]$, and $g[\varrho]$-Lipschitz continuous, and

$$(1.3) \qquad |\dot{f}| \leqq At^{-1+\varepsilon}, \qquad \varepsilon > 0, \qquad |g| \leqq At^{-1/2}.$$

The nature of the boundary operator Γ along the given arc AC is best described in terms of an auxiliary boundary value problem. Let D be the domain of points P lying to the left of AC, and let v be a differentiable function along AC. Γ is required to be such that the problem for $w(x, t) = w(P)$, $P \in D$,

$$
\begin{array}{llll}
 & \text{DE} & Lw = 0, & P \in D, \\
(1.4) & \text{BC} & \Gamma(w + v) = 0 & \text{on arc } AC, \\
 & \text{IC} & w(x, 0) = 0 & x < a, \\
 & \infty\text{C} & |w(x, t)| \leqq A, & \\
 & & |\sqrt{t}\, w_x(x, t| \leqq A, &
\end{array}
$$

has a unique solution.

In the following we shall derive a functional equation which the unknown boundary function $R(t)$ must satisfy. In the process of the derivation we shall also obtain a representation of $u(x, t)$ involving the boundary function. Thus the problem will be solved once the functional equation for the boundary is solved.

It is remarkable that in a problem of the type considered, where the notions of the boundary and of the solutions of the differential equations are interlocked, it is eventually possible to eliminate all reference to the latter. Due to this *reduction*, the onus of the problem is transposed to the solving of the equation for the boundary. While this need not be a trifle—this equation turns out to be a non-linear integro-differential equation—we shall here adopt the attitude that an essential simplification has been achieved, and that we are faced with a problem of a lesser order of difficulty.

In chapters to follow, we shall study and solve some of these equations. Already in Section 6 of this chapter, it will become apparent that the equations obtained in cases of physical interest—complicated as they are—are all of Volterra type and of the *second kind*. Hence they are amenable to numerical treatment. Further analysis will, of course, be required, if the solution in the large (or bounds for it) are desired.

We have here formulated the simplest free boundary value problem; namely the problem in which the medium is described by a single, one-dimensional heat equation and which is bounded by a free boundary on one side. From the context of Section 2, it will become apparent that various extensions can be easily achieved. One such extension—to two media each governed by a different heat equation and separated by the free boundary—is considered in full detail in Section 7, while others, also briefly discussed in that Section, will form the object of further studies.

Special problems of type (1.1) have been considered for over a century[2], although the problem never received as much attention as the corresponding free boundary problem for the potential equation. Except for the very special cases when the solution can be found explicitly, the only rigorous solutions can be found in [2, 13, 14, 3, 10, 11, 12]. The last three of these references are based already on the method proposed here. The demand for solutions resulted in the development of various approximation procedures which apply to selected cases with varying success. For samples of these consult [4, 5, 8].

2. Heuristic Considerations

Since the ordinary boundary value problem is usually considered of lesser difficulty than the free boundary problem, one is tempted to give up one of the free boundary conditions, and first solve the boundary value problem so posed with a ***fixed though arbitrary boundary***; then to use this solution for the determination of the boundary function by requiring that the second free boundary condition also be satisfied. Let $\varrho(t)$ be an arbitrary function satisfying condition (1.2), and denote by D_ϱ the domain formed analogously to D_R (see Figure 1), with arc AB represented by $x = \varrho(t)$. The first part of the proposed program is to find the functional $u[\varrho, f](P)$, $P \in D_\varrho$ satisfying

$$
\text{(2.1)}\quad
\begin{array}{lll}
\text{DE} & Lu = 0, & P \in D_\varrho, \\
\text{BC} & \Gamma u = 0 & \text{on arc } AC, \\
\text{BC} & u(\varrho, t) = f(t) & \text{for } 0 \leqq t \leqq \bar{t}.
\end{array}
$$

[2]For references that appeared prior to 1929 consult the expository paper of Brillouin [1].

The second step is to select among all functions $\varrho(t)$ those for which $u[\varrho, f]$ satisfies the second free boundary condition. That is, to solve the functional equation for ϱ,

$$u_x[\varrho, f](\varrho, t) = g(t). \tag{2.2}$$

This program is defeated at the first step, since it is impossible to determine the functional $u[\varrho, f]$ in closed form. Consequently, equation (4.2) cannot be written out.[3]

A successful attack stems from a different interpretation of the solution. Denote by D_R^+ the domain sought (heretofore denoted by D_R), and let D_R^- be the complementary domain, $P \in D_R^-[P: x < R(t),\ 0 < t \leqq \bar{t}]$. Extend the notion of the solution into the domain D_R^- by defining

$$u(x, t) \equiv 0, \qquad P \in D_R^-. \tag{2.3}$$

With this definition, $u(x, t)$ has the following properties:

$$\begin{array}{ll} Lu = 0, & P \in D_R^- \cup D_R^+, \\ \Gamma u = 0 & \text{on arc } AC, \\ u(x, 0) = 0 & \text{for } x < a, \\ [u] = f(t), & 0 \leqq t \leqq \bar{t}, \\ [u_x] = g(t), & 0 < t \leqq \bar{t}. \end{array} \tag{2.4}$$

Here [ ] denotes the jump across the line $x = R(t)$, i.e.,

$$[u] = \lim_{\substack{x \to R(t) \\ P \in D_R^+}} u(x, t) - \lim_{\substack{x \to R(t) \\ P \in D_R^-}} u(x, t), \qquad t \text{ fixed.}$$

This suggests that we seek the solution among all functions which satisfy, in addition to the first three conditions (2.4), the two jump conditions across an arbitrary curve $x = \varrho(t)$.

Let D_ϱ^- and D_ϱ^+ be domains formed as D_R^-, D_R^+, with $x = \varrho(t)$ replacing the separation line. The auxiliary functional $u^\varrho(x, t)$ is defined by

$$\begin{array}{lll} \text{DE} & Lu = 0, & P \in D_\varrho^- \cup D_\varrho^+, \\ \text{BC} & \Gamma u = 0 & \text{on arc } AC, \\ \text{IC} & u^\varrho(x, 0) = 0 & \text{for } x < a, \\ \text{JC1} & [u^\varrho] = f[\varrho] & \text{for } 0 \leqq t \leqq \bar{t}, \\ \text{JC2} & [u_x^\varrho] = g[\varrho] & \text{for } 0 < t \leqq \bar{t}. \end{array} \tag{2.5}$$

[3] An approximation procedure based on this program is, nevertheless, frequently used. In this procedure, one selects the zero-th approximation R_0 to R arbitrarily and solves (2.1) with $\varrho = R_0$ to get the first approximation u_1 to u. R_1 is now obtained by solving (2.2) with ϱ replaced by R_1, and higher order approximations are determined by reiterating this process. In applications one takes $R_0 \equiv a$. Usually then u_1 and R_1 can be determined explicitly, while the continuation presents already all the difficulties of the program. The sequences u_n, R_n, were shown in [3] to converge to the unique solution u, R of the problem in a special case.

We assume that ϱ is subject to condition (1.2), and that $\dot{\varrho}$ has a finite number of extremum points. Since $g[\varrho]$ may be singular at $t = 0$, we cannot expect u^ϱ to have a bounded x derivative in $D_\varrho^- \cup D_\varrho^+$. Some restrictive conditions are nevertheless required if the auxiliary functional is to be uniquely determined.

Let $\bar{D}_\varrho^\pm$ be the closure of $D_\varrho^\pm$ in the set $\bar{E}[-\infty < x < \infty,\ 0 \leqq t \leqq \bar{t},\ |x - a| + |t| > 0]$. Let $\bar{D} = \bar{D}_\varrho^+ \cup \bar{D}_\varrho^-$, and let D be the interior of $\bar{D}$. It is observed that both D and $\bar{D}$ are independent of $\varrho(t)$. Define $u^{\varrho\pm}(P)$, $P \in \bar{D}_\varrho^\pm$ by

$$u^{\varrho\pm}(P) = \lim_{Q \to P} u^\varrho(Q), \qquad Q \in D_\varrho^\pm. \tag{2.6}$$

We impose in addition to the requirement of existence of $u^{\varrho\pm}$ and $u_x^{\varrho\pm}$ the following conditions:

$$\begin{aligned} &|u^{\varrho\pm}| \leqq A \\ &|\sqrt{t}\, u_x^{\varrho\pm}| \leqq A \\ &u_x^\varrho(-\infty, t) = 0. \end{aligned} \tag{2.7}$$

It will be shown in Section 4 that the problem for the auxiliary functional $u^{\varrho\pm}$ has a *unique solution*. This solution will be determined *explicitly* provided that we know how to solve a certain boundary value problem in the *determined* domain D. Granted the uniqueness and existence of $u^{\varrho\pm}$, we have the following

THEOREM 1: *If the free boundary problem has a solution $u(P)$, $R(t)$, $P \in D_R^+$, then*

$$u = u^R, \qquad P \in D_R^+$$

and $R(t)$ satisfies

$$u^{R+}(R, t) = f[R], \qquad R(0) = a, \tag{2.8}$$

$$u_x^{R+}(R, t) = g[R], \qquad R(0) = a. \tag{2.9}$$

The truth of this statement follows from our reinterpretation of the free boundary value problem, and from uniqueness of the auxiliary functional.

THEOREM 2 (UNIQUENESS THEOREM). *If either one of the equations (2.8), (2.9) has at most one solution, then the free boundary value problem has at most one solution.*

Indeed, for every $R(t)$, $u(x, t)$ solving the free boundary problem, $R(t)$ must satisfy both equations.

Although we have thus established that $R(t)$ must satisfy both equation (2.8), (2.9), what is actually desired is to show that the solution of either one of these equations will also satisfy the other; for it would then be possible to prove the existence theorem for the solution of the original problem. This

will be shown in Section 5. Here we remark that if u^ϱ and u^ϱ_x were bounded, we would have for R satisfying (2.8),

$$u^{R-} = 0 \qquad \text{on the boundary of } D^-_R,$$
$$Lu^{R-} = 0, \qquad P \in D^-_R.$$

Hence, by a well known theorem, see e.g., [6], Chapter XXIX

$$u^{R-} \equiv 0, \qquad P \in D^-_R.$$

Consequently,

$$u^{R-}_x \equiv 0,$$

and

$$u^{R+}_x(R, t) = g[R] + u^{R-}_x(R, t) = g[R],$$

which is equation (2.9). Actually u^ϱ_x need not be bounded, but as we shall see, its behavior is sufficiently mild to enable us to prove that $u^{R-} \equiv 0$, $P \in D^-_R$, is a consequence of (2.8).

The construction of the auxiliary functional is based on properties of integrals analogous to the single and double layer potentials of potential theory which we now proceed to study.

3. Integrals Analogous to Simple and Double Layer Potentials

In this and in part of the next section we refer to the set E, the interior of the set $\bar{E}$ defined previously and to sets $E^\pm_\varrho$. E^-_ϱ is the same as D^-_ϱ, while E^+_ϱ is the set of points lying to the right of $x = \varrho(t)$. Let $U(x, \xi, t, \tau)$ be the fundamental solution of the heat equation,

$$(3.1) \qquad U(x, \xi, t, \tau) = \begin{cases} \dfrac{1}{2\sqrt{\pi(t-\tau)}} \exp\{-(x-\xi)^2/4(t-\tau)\}, & t > \tau \\ 0, & t \leqq \tau. \end{cases}$$

The simple and double layer potentials are defined by

$$(3.2) \qquad \begin{aligned} S^\varrho_{a,b}[h](x, t) &= \int_a^b U(x, \varrho(\tau), t, \tau) h(\tau)\, d\tau, \\ D^\varrho_{a,b}[x^h](x, t) &= \int_a^b U_\xi(x, \varrho(\tau), t, \tau) h(\tau)\, d\tau. \end{aligned}$$

If $a = 0$, $b = t$, we use the simplified notation

$$S_{0,t} = S, \qquad D_{0,t} = D.$$

$S^{\varrho\pm}$, $D^{\varrho\pm}$ are defined in $\bar{E}^\pm$ similarly to $u^{\varrho\pm}$.

These potentials have the following properties:

1) If $\varrho \in C[0, t]$, $|h| \leqq A$ in the half open interval $(0, t]$, and h is integrable, then $S^\varrho[h]$ is a regular solution of the heat equation in $E^-_\varrho \cup E^+_\varrho$ and is continuous in E. Furthermore, $S^\varrho[h](\varrho, t)$ has a meaning and

$$S^{\varrho\pm}[h](\varrho(t), t) = S^\varrho[h](\varrho, t).$$

2) If $\dot{\varrho} \in C(0, t]$, and h is Lipschitz continuous in the half open interval

$(0, t]$ and is also integrable, then $D^\varrho[h]$ is a regular solution of the heat equation in $E_\varrho^- \cup E_\varrho^+$ but has a jump of magnitude $h(t)$ across the line $x = \varrho(t)$. Furthermore the $D^{\varrho\pm}$ have a meaning in $\bar{E}_\varrho^\pm$, $D^\varrho[h](\varrho(t), t)$ has a meaning for $t > 0$, and

$$D^{\varrho\pm}[h](\varrho(t), t) = D^\varrho[h](\varrho, t) \pm \tfrac{1}{2}h(t), \qquad t > 0.$$

3) If ϱ and h satisfy the conditions of 2, then

$$S_x^\varrho[h] = -D^\varrho[h].$$

4) If $\dot{\varrho} \in C(0, t]$, and $\dot{h}$ exists in $(0, t]$, and is integrable, then

$$D_x^\varrho[h] = -\frac{h(0)}{2\sqrt{\pi t}} \exp\{-z^2(x, 0)\} - D^\varrho[h\dot{\varrho}] - S^\varrho[\dot{h}].$$

Here

$$z(x, \tau) = \frac{x - \varrho(\tau)}{2\sqrt{t - \tau}}; \quad z(x, 0) = \frac{x - \varrho(0)}{2\sqrt{t}}.$$

Properties of simple and double layers were established by Holmgren [7] in the case when the conditions imposed on ϱ and h held in the closed interval $[0, t]$. Our extension is based on the observation that for any $\delta > 0$,

$$S = S_{0,\delta} + S_{\delta,t},$$
$$D = D_{0,\delta} + D_{\delta,t}.$$

Both $S_{0,\delta}$ and $D_{0,\delta}$ are solutions of the heat equation *analytic* in x and t for $t > \delta$, while in the interval $[\delta, t]$ Holmgren's conditions are satisfied.

The formula for S_x^ϱ is obtained by differentiation under the integral sign. To get the formula for D_x^ϱ, we first write

$$D^\varrho[h] = D^\varrho[h] - S^\varrho[h\dot{\varrho}] + S^\varrho[h\dot{\varrho}].$$

Now

$$D^\varrho[h] - S^\varrho[h\dot{\varrho}] = \frac{1}{2\sqrt{\pi}} \int_0^t \left(\frac{z(x, \tau)}{t - \tau} - \frac{\dot{\varrho}(\tau)}{\sqrt{t - \tau}}\right) \exp\{-z^2(x, \tau)\} h(\tau)\, d\tau$$
$$= \frac{1}{\sqrt{\pi}} \int_0^t z_\tau(x, \tau) \exp\{-z^2(x, \tau)\} h(\tau)\, d\tau$$
$$= -\frac{h(0)}{\sqrt{\pi}} \int_{z(x,t)}^{z(x,0)} \exp\{-\sigma^2\}\, d\sigma - \frac{1}{\sqrt{\pi}} \int_0^t \dot{h}(\tau) \left(\int_{z(x,t)}^{z(x,\tau)} \exp\{-\sigma^2\}\, d\sigma\right) d\tau,$$

on integrating by parts. Here

$$z(x, t) = \begin{cases} +\infty & \text{if } x > \varrho(t) \\ -\infty & \text{if } x < \varrho(t). \end{cases} \tag{3.4}$$

Differentiation and use of the formula for S_x^ϱ now yield the result in the desired form.

4. Construction of the Auxiliary Functional

We first consider a special case when D_ϱ^+ is E_ϱ^+, that is when the given arc AC consists of the x-axis for $x > a$, and when Γ is the identity operator. The solution in this case will be referred to as the fundamental part v^ϱ. It is required to satisfy equations (2.5)–(2.7) with the exception of BC in (2.5) which is replaced by

$$\text{(4.1)} \qquad \text{BC} \quad \begin{aligned} &v^\varrho(x, 0) = 0, \qquad x > a, \\ &v^\varrho(\infty, t) = 0. \end{aligned}$$

We will now show that

$$\text{(4.2)} \qquad v^\varrho = D^\varrho[f] - S^\varrho[g + f\dot{\varrho}]$$

is the *unique solution* of this problem.

First recall the conditions imposed on ϱ, f and g, namely:

$$\text{(4.3)} \qquad \begin{aligned} &\varrho \in C_1(0, \bar{t}], \qquad \dot{\varrho} \text{ has at most a finite number of extremum points,} \\ &f \in C_1(0, \bar{t}], \\ &g \text{ is Lipschitz continuous in } (0, \bar{t}], \\ &|\dot{\varrho}| \leq At^{-1/2}, \qquad \varrho(0) = a, \\ &|\dot{f}| \leq At^{-1+\varepsilon}, \qquad \varepsilon > 0, \\ &|g| \leq At^{-1/2}. \end{aligned}$$

Here, and thereafter, A denotes a positive constant, not always the same.

1. ***v satisfies the differential equations in $E_\varrho^- \cup E_\varrho^+$.***

This follows from properties 1 and 2, discussed in the previous section, since the conditions on ϱ and $h = f$ in the case of the double layer, and on ϱ and $h = g + f\dot{\varrho}$ in the case of the single layer, are satisfied.

2. ***v^ϱ satisfies the jump conditions.***

Using properties 1 and 2, $v^{\varrho\pm}$ is defined, and

$$v^{\varrho\pm}(\varrho, t) = v^\varrho(\varrho, t) \pm \tfrac{1}{2}f(t),$$
$$[v^\varrho] = f(t).$$

Now, using the differentiation formulae (properties 3 and 4) we have

$$v_x^\varrho = -\frac{f(0)}{2\sqrt{\pi t}} \exp\{-z^2(x, 0)\} + D^\varrho[g] - S^\varrho[\dot{f}].$$

Again, the conditions on ϱ and $h = g$ in the case of the double layer, and on ϱ and $h = f$ in the case of the single layer are satisfied; by properties 1 and 2, $v_x^{\varrho\pm}$ exists and

$$v_x^{\varrho\pm}(\varrho, t) = v_x^{\varrho}(\varrho, t) \pm \tfrac{1}{2}g(t),$$
$$[v_x^{\varrho}] = g(t).$$

3. $v^{\varrho}(x, 0) = 0$ for $x \neq a$.

Using the definitions of S^{ϱ} and D^{ϱ}, equations (3.2),

$$v^{\varrho} = \frac{1}{2\sqrt{\pi}} \int_0^t \left(\frac{f(\tau)}{t-\tau} z(x, \tau) - \frac{g(\tau) + f(\tau)\dot{\varrho}(\tau)}{\sqrt{t-\tau}}\right) \exp\{-z^2(x, \tau)\}\, d\tau.$$

Let $\delta > 0$, $0 \leqq t < \delta$. Then $|\varrho(t) - a| \leqq 2A\sqrt{\delta}$, as seen from conditions (4.3). Assume that

$$|x - a| \geqq 2A\sqrt{\delta} + 2\varepsilon, \qquad \varepsilon > 0.$$

We get, for finite x,

$$0 < 2\varepsilon \leqq |x - a| - 2A\sqrt{\delta} \leqq |x - \varrho(t)|$$
$$= |(x - a) - (\varrho(t) - a)| \leqq |x - a| + 2A\sqrt{\delta} \leqq A,$$

if $0 \leqq t < \delta$. Therefore, using (4.3)

$$|f(\tau)z(x, \tau) \exp\{-z^2(x, \tau)\}| \leqq \frac{A}{\sqrt{t-\tau}} \exp\{-(\varepsilon/\sqrt{t-\tau})^2\},$$

$$|g(\tau) + f(\tau)\dot{\varrho}(\tau)| \exp\{-z^2(x, \tau)\} \leqq \frac{A}{\sqrt{\tau}} \exp\{-(\varepsilon/\sqrt{t-\tau})^2\}.$$

Hence

$$|v^{\varrho}| \leqq A \int_0^t \left(\frac{1}{(t-\tau)^{3/2}} + \frac{1}{\sqrt{\tau}\sqrt{t-\tau}}\right) \exp\{-(\varepsilon/\sqrt{t-\tau})^2\}\, d\tau$$
$$= 2\left(\frac{1}{\varepsilon} + \sqrt{\pi}\right) A \int_{\varepsilon/\sqrt{t}}^{\infty} \exp\{-\sigma^2\}\, d\sigma.$$

This shows that $|v^{\varrho}| \to 0$ as $t \to 0$ and $x \neq a$ since δ and ε are arbitrary.

4. $v^{\varrho}(\pm\infty, t) = 0$.

Let $|x - a| > 2A\sqrt{t}$, for any t. Then

$$|x - \varrho(\tau)| \geqq |x - a| - 2A\sqrt{t} = k \qquad \text{for } 0 \leqq \tau \leqq t$$

and $k \to \infty$ as $|x| \to \infty$. Now

$$\frac{1}{\sqrt{t-\tau}} = \frac{2}{x - \varrho(\tau)} z(x, \tau) \leqq \frac{2}{k} z(x, \tau)$$
$$\frac{1}{t-\tau} \leqq \frac{4}{k^2} z^2(x, \tau).$$

Also, for any value that z may assume,

$$|z \exp\{-z^2\}| \leqq \frac{1}{\sqrt{2e}},$$

$$|z^3 \exp\{-z^2\}| \leqq \left(\frac{3}{2e}\right)^{3/2}.$$

Therefore,

$$\left|\frac{f(\tau)}{t-\tau} z \exp\{-z^2\}\right| \leqq \frac{A}{k^2}|z^3 \exp\{-z^2\}| \leqq \frac{A}{k^2}$$

$$\left|\frac{g(\tau)+f(\tau)\dot{\varrho}(\tau)}{\sqrt{t-\tau}} \exp\{-z^2\}\right| \leqq \frac{A}{\sqrt{\tau}k}|z \exp\{-z^2\}| \leqq \frac{A}{\sqrt{\tau}k},$$

and

$$|v^\varrho| \leqq A \int_0^t \left(\frac{1}{k^2}+\frac{1}{k\sqrt{\tau}}\right) d\tau \to 0 \quad \text{as } k \to \infty.$$

5. $|v^\varrho| \leqq A$.

Observing that

$$\frac{1}{2}\left(\frac{z(x,\tau)}{t-\tau} - \frac{\dot{\varrho}(\tau)}{\sqrt{t-\tau}}\right) = z_\tau(x,\tau),$$

we get on integrating by parts,

$$v^\varrho = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{g(\tau)}{\sqrt{t-\tau}} \exp\{-z^2(x,\tau)\}\, d\tau - \frac{f(0)}{\sqrt{\pi}} \int_{z(x,t)}^{z(x,0)} \exp\{-\sigma^2\}\, d\sigma$$
$$- \frac{1}{\sqrt{\pi}} \int_0^t \dot{f}(\tau)\left(\int_{z(x,t)}^{z(x,\tau)} \exp\{-\sigma^2\}\, d\sigma\right) d\tau.$$

The last two integrals are clearly finite, since $|\dot{f}| \leqq At^{-1+\varepsilon}$. As to the first integral, we have

$$\left|\frac{g(\tau)}{\sqrt{t-\tau}} \exp\{-z^2(x,\tau)\}\right| \leqq \frac{A}{\sqrt{\tau}\sqrt{t-\tau}},$$

so that

$$\left|\frac{1}{2\sqrt{\pi}} \int_0^t \frac{g(\tau)}{\sqrt{t-\tau}} \exp\{-z^2(x,\tau)\}\, d\tau\right| \leqq \frac{A}{2\sqrt{\pi}} \int_0^t \frac{d\tau}{\sqrt{\tau}\sqrt{t-\tau}} = \frac{A\sqrt{\pi}}{2}.$$

Hence, v^ϱ is finite.

6. $|\sqrt{t}v_x^\varrho| \leqq A$.

Using the definitions of S^ϱ, D^ϱ, we have, from the formula for v_x^ϱ

$$v_x^\varrho = -\frac{f(0)}{2\sqrt{\pi t}} \exp\{-z^2(x,0)\} - \frac{1}{2\sqrt{\pi}} \int_0^t \frac{\dot{f}(\tau)}{\sqrt{t-\tau}} \exp\{-z^2(x,\tau)\, d\tau$$
$$+ \frac{1}{2\sqrt{\pi}} \int_0^t \frac{g(\tau)}{t-\tau} z(x,\tau) \exp\{-z^2(x,\tau)\}\, d\tau.$$

Denoting the three terms by α, β, γ, consecutively, we have

$$|\alpha| \leqq At^{-1/2};$$

$$|\beta| \leqq A \int_0^t \frac{d\tau}{\tau^{-1+\delta}\sqrt{t-\tau}} = At^{-1/2} \int_0^1 \frac{d\sigma}{\sigma^{-1+\delta}\sqrt{1-\sigma}} \leqq At^{-1/2}, \qquad t \leqq \bar{t}.$$

To estimate γ we split the interval of integration into two parts: from 0 to $t/2$ and from $t/2$ to t. For $0 \leqq \tau \leqq t/2$,,

$$\left| \frac{g(\tau)}{t-\tau} z \exp\{-z^2\} \right| \leqq \frac{A}{t\sqrt{\tau}}.$$

For $t/2 < \tau < t$,

$$|g(\tau)| \leqq At^{-1/2}$$

$$\left| \frac{z}{2(t-\tau)} \exp\{-z^2\} \right| \leqq \left| \frac{z}{2(t-\tau)} - \frac{\dot{\varrho}(\tau)}{2\sqrt{t-\tau}} \right| . \exp\{-z^2\}$$

$$+ \left| \frac{\dot{\varrho}(\tau)}{2\sqrt{t-\tau}} \right| \exp\{-z^2\} \leqq |z_\tau(x,\tau)| \exp\{-z^2(x,\tau)\} + \frac{A}{\sqrt{\tau}\sqrt{t-\tau}}.$$

Hence

$$|\gamma| \leqq \frac{A}{t} \int_0^{t/2} \frac{d\tau}{\sqrt{\tau}} + \frac{A}{\sqrt{t}} \int_{t/2}^t \frac{d\tau}{\sqrt{\tau}\sqrt{t-\tau}} + \frac{A}{\sqrt{t}} \int_{t/2}^t \exp\{-z^2(x,\tau)\} |z_\tau(x,\tau)| \, d\tau.$$

Now, since $\dot{\varrho}(t)$ has at most a finite number of extremum points, the same holds for $\varrho(t)$, and consequently

$$z_\tau(x,\tau) = \frac{1}{2\sqrt{t-\tau}} \left(\frac{x-\varrho(\tau)}{2(t-\tau)} - \dot{\varrho}(\tau) \right)$$

may have at most a finite number of zeros, τ_i, as τ runs from $t/2$ to t. Therefore,

$$\int_{t/2}^t \exp\{-z^2(x,z)\} |z_\tau(x,\tau)| \, d\tau = \left| \int_{t/2}^{\tau_1} \right| + \sum_{i=1}^{n-1} \left| \int_{\tau_i}^{\tau_{i+1}} \right| + \left| \int_{\tau_n}^t \right| \leqq \sqrt{\pi}(n+1)$$

where n is the number of zeros. We now get

$$|\gamma| \leqq \frac{A}{\sqrt{t}} \int_0^{1/2} \frac{d\sigma}{\sqrt{\sigma}} + \frac{A}{\sqrt{t}} \int_{1/2}^1 \frac{d\sigma}{\sqrt{\sigma(1-\sigma)}} + \frac{A\sqrt{\pi}(n+1)}{\sqrt{t}} \leqq \frac{A}{\sqrt{t}}.$$

Adding the three results,

$$|\sqrt{t}v_x^\varrho| \leqq \sqrt{t}(|\alpha| + |\beta| + |\gamma|) \leqq A.$$

This completes the proof of the existence theorem. We now show that

7. *v^ϱ is the unique solution for the auxiliary functional in $E_\varrho^- \cup E_\varrho^+$.*

Suppose that to the contrary there are two such solutions v_1^ϱ and v_2^ϱ and let

$$\overline{v} = v_1^\varrho - v_2^\varrho.$$

$\overline{v}$ satisfies the same conditions as v^ϱ, except that both $\overline{v}$ and $\overline{v}_x$ are continuous in E.

We use the identity

$$2 \iint\limits_{\text{domain}} \varphi_x^2 dx dt = \oint\limits_{\text{boundary}} \varphi^2 dx + 2\varphi\varphi_x dt \tag{4.4}$$

satisfied by regular solutions of the heat equation, see e.g. [6], which we apply to $\overline{v}$ in the domain indicated on Figure 2. This domain is obtained from the rectangle $|x| \leqq k$, large k, $0 \leqq t \leqq t_0$, $t_0 \leqq \bar{t}$ but otherwise arbitrary, by cutting out a strip of width δ about the curve $x = \varrho(t)$ and a rectangle of width $2A\sqrt{\varepsilon}$ and height ε centered at $x = \varrho(0)$, $t = 0$. We now let $\delta \to 0$

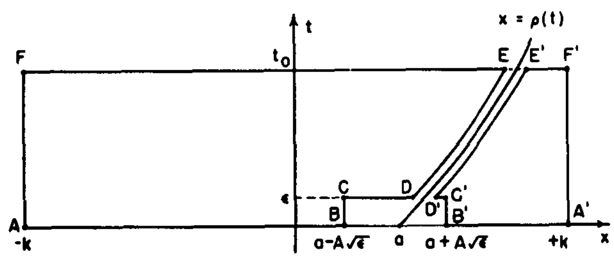

Figure 2

and $k \to \infty$, keeping ε fixed. Contributions to the line integral along AF and $A'F'$ vanish, while those along DE and $D'E'$ cancel each other. Furthermore, on AB, $A'B'$, $\overline{v} = 0$, so that also there the contribution is nil. Along BC and $B'C'$ the contribution is bounded by

$$2 \int_0^\varepsilon \frac{A}{\sqrt{t}} dt = 4A\sqrt{\varepsilon},$$

while along CD and $C'D'$ we get at most

$$\int_{a-A\sqrt{\varepsilon}}^{a+A\sqrt{\varepsilon}} A^2 dx = 2A^3\sqrt{\varepsilon},$$

see conditions (2.7). Hence these also vanish as $\varepsilon \to 0$. Passing to the limit, we now get

$$2 \iint\limits_{0 \leq t \leq t_0} \overline{v}_x^2 dx dt + \int_{-\infty}^{\infty} \overline{v}^2(x, t_0)\, dx = 0,$$

i.e., $\bar{v}(x, t_0) = 0$. Consequently $\bar{v}(x, t) \equiv 0$, and $v_1^\varrho \equiv v_2^\varrho = v^\varrho$.

We now consider the general problem for the auxiliary functional. Write u^ϱ as the sum of the fundamental part v^ϱ and a complementary part w^ϱ,

$$u^\varrho = v^\varrho + w^\varrho, \qquad P \in D_\varrho^- \cup D_\varrho^+. \tag{4.5}$$

Substituting in (2.5) and (2.7) and using the properties of v^ϱ we find that w^ϱ satisfies

$$\begin{array}{llll} & \text{DE} & Lw^\varrho = 0, & P \in D, \\ & \text{BC} & \Gamma(w^\varrho + v^\varrho) = 0 & \text{on arc } AC, \text{ Figure 1}, \\ (4.6) & \text{IC} & w^\varrho(x, 0) = 0 & \text{for } x < \varrho(0) = a, \\ & \infty\text{C} & |w^\varrho(x, t)| \leqq A, & \\ & & |\sqrt{t} w^\varrho(x, t)| \leqq A & \end{array}$$

v^ϱ is clearly continuously differentiable on AC, and therefore, by the assumption on the nature of the boundary operator Γ, see Section 1, equations (4.6) have a *unique solution.* u^ϱ is also unique; for, if there were two such solutions, u_1^ϱ and u_2^ϱ, then

$$w_1^\varrho = u_1^\varrho - v^\varrho,$$
$$w_2^\varrho = u_2^\varrho - v^\varrho$$

would both satisfy equations (4.6) and consequently we would have

$$w_1^\varrho \equiv w_2^\varrho \equiv w^\varrho, \quad u_1^\varrho \equiv u_2^\varrho \equiv u^\varrho.$$

5. Reduction Theorems

It will now be shown that the free boundary problem can be reduced to solving a certain functional equation for the boundary. Consider the following equations

$$u^{\varrho-}(\varrho, t) = 0, \qquad \varrho(0) = a, \tag{5.1}$$

$$u_x^{\varrho-}(\varrho, t) = 0, \qquad \varrho(0) = a, \tag{5.2}$$

$$\dot{\varrho}(t) u^{\varrho-}(\varrho, t) + 2u_x^{\varrho-}(\varrho, t) = 0, \qquad \varrho(0) = a. \tag{5.3}$$

Explicitly, these equations are

$$(5.1)' \quad v^\varrho(\varrho, t) - \tfrac{1}{2}f[\varrho] + w^\varrho(\varrho, t) = D^\varrho[f](\varrho, t) - S^\varrho[g + f\dot{\varrho}](\varrho, t) - \tfrac{1}{2}f[\varrho] + w^\varrho(\varrho, t) = 0,$$

$$(5.2)' \quad v_x^\varrho(\varrho, t) - \tfrac{1}{2}g[\varrho] + w_x^\varrho(\varrho, t)$$
$$= -\frac{f(\varrho(0))}{2\sqrt{\pi t}} \exp\{-z^2(\varrho(t), 0)\} + D^\varrho[g](\varrho, t) - S^\varrho[\dot{f}](\varrho, t) - \tfrac{1}{2}g[\varrho] + w_x^\varrho(\varrho, t) = 0,$$

$$(5.3)' \quad \dot{\varrho}(t) v^\varrho(\varrho, t) + 2v_x^\varrho(\varrho, t) - \tfrac{1}{2}(\dot{\varrho}(t) f(t) + 2g(t)) + \dot{\varrho} w^\varrho(\varrho, t) + 2w_x^\varrho(\varrho, t) = 0.$$

Assume that these equations have solutions R_1, R_2, R_3 respectively which satisfy the conditions imposed on $\varrho(t)$, (see conditions (4.3)).

THEOREM 3. *If $R_1(t)$ exists for $0 \leqq t \leqq \bar{t}$, then*

$$R(t) = R_1(t),$$
$$u(x, t) = u^{R_1^+}(x, t), \qquad P \epsilon D^+_{R_1}$$

form a solution of the free boundary problem.

Proof: Consider $u^{R_1^-}(x, t)$, $P \epsilon D^-_{R_1}$, and apply the identity (4.4) to $u^{R_1^-}$ in the domain $ABCDEFA$, Figure 2, with $\varrho(t)$ replaced by $R_1(t)$ and $\delta = 0$. Along AB the contribution is zero, since $u^{\varrho}(x, 0) = 0$ for $x < a$. Along DE the contribution is zero in view of the definition of R_1. Contributions along AF and BCD tend to zero as one lets $k \to \infty$, $\varepsilon \to 0$, and one gets

$$2\iint\limits_{D^-_R} (u_x^{R_1^-})^2\, dxdt + \int_{-\infty}^{R_1(t)} (u^{R_1^-})^2\, dx = 0.$$

Hence

$$u^{R_1^-}(x, t) \equiv 0, \qquad P \epsilon D^-_{R_1},$$
$$u_x^{R_1^-}(R_1, t) \equiv 0.$$

Since all conditions imposed on $u(x, t)$ are automatically satisfied by $u^{\varrho}(x, t)$ for any ϱ, with the exception of the free boundary conditions, we have to verify the latter. Now

$$u^{R_1^+}(R_1, t) = u^{R_1^-}(R_1, t) + f[R_1] = f[R_1],$$
$$u_x^{R_1^+}(R_1, t) = u_x^{R_1^-}(R_1, t) + g[R] = g[R],$$

completing the proof.

THEOREM 4. *If $R_2(t)$ exists for $0 \leqq t \leqq \bar{t}$, and $\dot{R}_2(t) \leqq 0$, then*

$$R(t) = R_2(t),$$
$$u(x, t) = u^{R_2^+}(x, t), \qquad P \epsilon D^+_{R_2}$$

form a solution of the free boundary problem.

Proof: Proceeding as in Theorem 3, we find

$$0 \leqq 2 \iint\limits_{D^-_{R_2}} (u_x^{R_2^-})^2\, dxdt + \int_{-\infty}^{R_2(t)} (u^{R_2^-})^2\, dx = \int_0^t (u^{R_2^-}(R_2, t))^2 \dot{R}_2(t)\, dt \leqq 0.$$

Hence

$$u^{R_3^-} \equiv 0, \qquad P \in D^-_{R_3}.$$

The remainder of the proof is identical with the preceding proof.

THEOREM 5. *If $R_3(t)$ exists for $0 \leqq t \leqq \bar{t}$, then*

$$R(t) = R_3(t),$$

$$u(x, t) = u^{R_3^+}(x, t), \qquad P \in D^+_{R_3}$$

form a solution of the free boundary problem.

Proof: One proceeds exactly as in the proof of Theorem 3.

Remark: Clearly, every solution of (5.1) is also a solution of (5.2) and (5.3), and every solution of (5.3) is also a solution of (5.1) and (5.2). As to solutions of (5.2), Theorem 4 in conjunction with Theorem 1 show that only those solutions of (5.2) which have a nonpositive derivative are also solutions of (5.1) and (5.3). Although we feel that in the latter case the sign of the derivative is immaterial, we were not able to prove Theorem 4 when this sign is positive. This is, unfortunately, a serious drawback. To determine R we have at our disposal three equations. (Many other equations can be also obtained.) In applications—see Section 6—it turns out that equation (5.2) is usually the easiest to handle. The existence of a solution R of this equation insures the existence of a solution to the free boundary problem only when $\dot{R} \leqq 0$. If it happens that $\dot{R} > 0$ for some t then the existence theorem remains unproven, and one has to consider either equation (5.1) or equation (5.3).

6. Applications

We consider a number of examples drawn from physics of change of phase. In each case we shall derive the auxiliary functional and the equation for the free boundary. Some of these equations will be considered in detail in the following papers.

We define

$$\eta(\sigma) = \frac{1}{\sqrt{\pi}} \exp\{-\sigma^2\}, \tag{6.1}$$

$$z(x, \tau) = \frac{x - \varrho(\tau)}{2\sqrt{t - \tau}}, \qquad 0 \leqq \tau \leqq t, \tag{6.2}$$

and observe that

$$z(x, t) = \lim_{\tau \to t} \frac{x - \varrho(\tau)}{2\sqrt{t - \tau}} = \begin{cases} +\infty & \text{if } x > \varrho(t) \\ 0 & \text{if } x = \varrho(t), \\ -\infty & \text{if } x < \varrho(t). \end{cases} \qquad t > 0 \tag{6.3}$$

1. *Motion of a plane liquid-vapor interface.*

A liquid in the presence of an undersaturated mixture containing its own vapor will evaporate, while if the mixture is supersaturated, the vapor will condense. We assume that the process occurs at a constant temperature and denote by g the saturation vapor density, and by c_0 the initial vapor density. The problem is to find the interface, $x = R(t)$, $R(0) = 0$, and the vapor density $u(x, t)$ satisfying

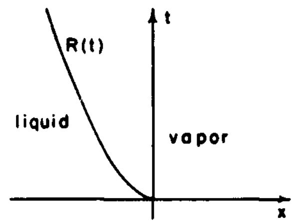

Figure 3

DE	$Du_{xx} = u_t$	for $x > R(t)$,	$t > 0$
BC	$u(x, 0) = c_0$	for $x \geqq 0$,	
FBC1	$u(R, t) = g$	for $t \geqq 0$.	

To these we adjoin a second boundary condition on the free boundary expressing the conservation of mass across the boundary, namely

$$\text{FBC2} \qquad Du_x(R, t) = (\varrho - g)\dot{R}(t) \qquad \text{for } t > 0.$$

Here D is the coefficient of diffusion, and ϱ the density of the liquid.

On introducing dimensionless variables and functions, $\bar{x}, \bar{t}$, $\bar{R}(\bar{t})$ and $\bar{u}(\bar{x}, \bar{t})$, defined by

$$x = l\bar{x}, \ t = l^2D^{-1}\bar{t}, \ \bar{u}(\bar{x}, \bar{t}) = \frac{u(x, t) - c_0}{g - c_0}, \ R(t) = l\bar{R}(\bar{t}),$$

the problem reduces, on omitting bars, to finding $u(x, t)$, $R(t)$ satisfying

DE	$u_{xx} = u_t$,	$x > R(t)$, $t > 0$,
BC	$u(x, 0) = 0$,	$x > R(0) = 0$,
FBC1	$u(R, t) = 1$,	$t \geqq 0$,
FBC2	$\alpha u_x(R, t) = \dot{R}$,	$t > 0$.

The specifications are completed by adding a condition at infinity which on physical grounds is $u(\infty, t) = 0$. In the above, l is an arbitrary length, while

$$\alpha = \frac{g - c_0}{\varrho - g}.$$

If the vapor is undersaturated, $c_0 < g$ and $\alpha > 0$. In the case of supersaturation, $c_0 > g$ and $\alpha < 0$. Since, however, $\varrho > c_0$, it follows that in this case

$$\alpha = -\frac{c_0 - g}{\varrho - g} > -1.$$

The equality $\alpha = -1$ occurs only when $c_0 = \varrho$, that is, when the liquid cannot be distinguished from its vapor.

Thus, this problem is of the type formulated in Section 1, with $f(t) \equiv 1$, $g(t) \equiv \alpha^{-1}\dot{R}(t)$, $a = 0$, while the given part of the boundary is the positive x-axis, and Γ is the identity operator. Part of the given boundary has been removed to infinity, and the problem would not be properly posed without some growth condition as $x \to \infty$. We chose here the conditions $u(\infty, 0) = 0$ but the result would be the same if we had only required that $|u(x, t)| \leqq A \exp\{kx^2\}$, $t < \infty$. Conditions (1.2), (1.3) on f and g are obviously satisfied.

This problem can be solved explicitly.[4] The result is

$$u = a \int_{\frac{x}{2\sqrt{t}}}^{\infty} \exp\{-\sigma^2\}\, d\sigma,$$

$$R(t) = -2b\sqrt{t}$$

where

$$a = \left(\int_{-b}^{\infty} \exp\{-\sigma^2\}\, d\sigma\right)^{-1}$$

and b is the root of

$$\alpha = 2b \cdot \exp\{b^2\} \int_{-b}^{\infty} \exp\{-\sigma^2\}\, d\sigma.$$

This equation for b has a unique real root provided $\alpha > -1$, and $b \to -\infty$ as $\alpha \to -1$. This does not, however, imply the uniqueness of the solution of the problem itself.

It is observed that in this problem, $R(t) = O(t^{-1/2})$. The same singularity will occur in some of the other problems considered here; this is at the root of our assumptions on the behavior of $f(t)$ and $g(t)$, see conditions (1.2), (1.3). Had we imposed more stringent conditions, these problems would be eliminated from the general analysis.

Although a solution is known, it is nevertheless instructive to apply the method discussed in this paper. Since in this example $D_\varrho^+ = E_\varrho^+$, and

$$\Gamma(u) = \begin{Bmatrix} u(x, 0) \\ u(\infty, t) \end{Bmatrix} = 0 \quad \begin{matrix} x > 0 \\ t \geqq 0 \end{matrix}$$

[4]From dimensional analysis one can conclude that $u(x, t)$ is a function of $x/\sqrt{t}$ only and the result follows at once.

the auxiliary functional u^ϱ consists of the fundamental part v^ϱ only. Using formula (4.2),

$$u^\varrho(x,t) = \tfrac{1}{2}\int_0^t \left(\frac{z(x,t)}{t-\tau} - \frac{\dot\varrho/\alpha + \dot\varrho(\tau)}{\sqrt{t-\tau}}\right)\eta(z(x,\tau))\,d\tau$$
$$= \int_{z(x,0)}^{z(x,t)} \eta(\sigma)d\sigma - \frac{1}{2\alpha}\int_0^t \frac{\dot\varrho(\tau)}{\sqrt{t-\tau}}\eta(z(x,\tau))\,d\tau.$$
$$u^\varrho_x(x,t) = -\frac{1}{2\sqrt{t}}\eta(z(x,0)) + \frac{1}{2\alpha}\int_0^t \frac{\dot\varrho(\tau)}{t-\tau}\eta(z(x,\tau))\,d\tau.$$

The equations (5.1)–(5.3) are in this case:

a) $$\int_{z(\varrho,0)}^0 \eta(\sigma)d\sigma - \frac{1}{2\alpha}\int_0^t \frac{\dot\varrho(\tau)}{\sqrt{t-\tau}}\eta(z(\varrho,\tau))d\tau - \tfrac{1}{2} = 0, \qquad \varrho(0) = 0,$$

b) $$-\frac{1}{2\sqrt{t}}\eta(z(\varrho,0)) + \frac{1}{2\alpha}\int_0^t \frac{\dot\varrho(\tau)}{t-\tau}z(\varrho,\tau)\eta(z(\varrho,\tau))d\tau - \frac{\dot\varrho(t)}{2\alpha} = 0, \quad \varrho(0) = 0,$$

c) $$\dot\varrho(t)\left(\int_{z(\varrho,0)}^0 \eta(\sigma)d\sigma - \frac{1}{2\alpha}\int_0^t \frac{\dot\varrho(\tau)}{\sqrt{t-\tau}}\eta(z(\varrho,\tau))d\tau\right) - \frac{1}{\sqrt{t}}\eta(z(\varrho,0))$$
$$+\frac{1}{\alpha}\int_0^t \frac{\dot\varrho(\tau)}{t-\tau}z(\varrho,\tau)\eta(z(\varrho,\tau))d\tau - \left(\frac{1}{2}+\frac{1}{\alpha}\right)\dot\varrho(t) = 0, \quad \varrho(0) = 0.$$

The first of these equations is of the first kind and would be difficult to handle. The second one is clearly of the form

b′) $$\dot\varrho(t) = G[\varrho], \quad \varrho(0) = 0,$$

suitable for an iterative attack. Its solution need not, however, imply the solution of the original problem, unless the result has a negative derivative; see Theorem 4. However, every solution of the problem must satisfy this equation, so that a statement concerning the uniqueness of solutions can be made independently of the sign of the derivative. We know, however, that the free boundary problem has a solution, while the uniqueness question is still open. This can now be resolved by showing that equation b) has a unique solution.

It is now possible to indicate in what sense the auxiliary functional u^ϱ differs from the functional $u[\varrho, f]$ defined in Section 2 by equations (4.1). Even in this simple case $u[\varrho, f]$ cannot be determined in closed form. If, however, we further restrict the class of functions ϱ to functions

$$\varrho = -2b\sqrt{t}, \qquad -\infty < b < \infty,$$

one finds

$$u[\varrho, f] = \left(\int_{x/2\sqrt{t}}^{\infty} \exp\{-\sigma^2\}\, d\sigma\right) \Big/ \left(\int_{-b}^{\infty} \exp\{-\sigma^2\}\, d\sigma\right), \quad P \in D_\varrho^+.$$

For u^ϱ, we get, after somewhat involved computations,

$$u^\varrho = \frac{1}{\sqrt{\pi}}\left(1 + \frac{2b}{\alpha}\exp\{b^2\}\int_b^{s(x,t)} \exp\{-\sigma^2\}d\sigma\right)\int_{x/2\sqrt{t}}^{s(x,t)} \exp\{-\sigma^2\}d\sigma, \; P \in D_\varrho^- \cup D_\varrho^+.$$

The two functionals are defined as functions of (x, t) over different domains. They are both defined in D_ϱ^+, and

$$u^{\varrho+}(x, t) \equiv u[\varrho, f](x, t)$$

only if

$$\frac{1}{\sqrt{\pi}}\left(1 + \frac{2b}{\alpha}\exp\{b^2\}\int_b^{\infty} \exp\{-\sigma^2\}\, d\sigma\right) = \left(\int_{-b}^{\infty} \exp\{-\sigma^2\}\, d\sigma\right)^{-1},$$

or, if b is such that

$$2b \exp\{b^2\} \int_{-b}^{\infty} \exp\{-\sigma^2\}\, d\sigma = \alpha.$$

For this value of b, however, $u^{\varrho-} \equiv 0$. Hence the only function which the two functionals have in common is the solution of the free boundary problem.

2. *Motion of plane liquid-vapor interface in a confined medium.*

If we complete the picture of example 1 with the requirement that the

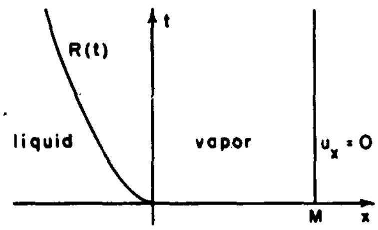

Figure 4

vapor remains confined in the domain $x \leqq M$, while at M there is an impervious boundary, see Figure 4, the problem is identical with the preceeding except that:

$u(x, t)$ is defined for $R(t) < x < M$, $t > 0$, $u_x(M, t) = 0$.

Here $f[\varrho] = 1$, $g[\varrho] = \dfrac{\dot{\varrho}}{\alpha}$, $D_\varrho^+[\varrho(t) < x < M]$, $a = 0$, and

$$\Gamma u = \begin{Bmatrix} u(x, 0) \\ u_x(M, t) \end{Bmatrix} = 0 \qquad \begin{matrix} \text{for } 0 < x < M \\ \text{for } \quad t \geqq 0. \end{matrix}$$

The fundamental part is the same as in example 1. The complementary part can be found by the image method, and one gets

$$w^{\varrho} = v^{\varrho}(2M - x, t), \qquad x \leqq M.$$

Thus,

$$u^{\varrho} = \int_{z(x,0)}^{z(x,t)} \eta(\sigma)\, d\sigma + \int_{z(2M-x,0)}^{\infty} \eta(\sigma) d\sigma - \frac{1}{2\alpha} \int_0^t \frac{\dot{\varrho}(\tau)}{\sqrt{t-\tau}} [\eta(z(x,\tau)) + \eta(z(2M - x, t))]\, d\tau.$$

Here again, equation (5.2) for the determination of $R(t)$ is easier to handle than the other two equations. This equation is

$$u_x^{\varrho -}(\varrho, t) = 0, \qquad \varrho(0) = 0,$$

or

$$\dot{\varrho}(t) = -\frac{\alpha}{\sqrt{t}} [\eta(x(\varrho, 0)) - \eta(z(2M - \varrho, 0))] + \int_0^t \frac{\dot{\varrho}(\tau)}{t-\tau} [z(\varrho, \tau)\eta(z(\varrho, \tau)) - z(2M - \varrho, \tau)\eta(z(2M - \varrho, \tau))]\, d\tau,$$

$\varrho(0) = 0.$

Again, the solution $R(t)$ of this equation will be the desired free boundary only if $\dot{R}(t)$ turns out to be negative and this is the case when $\alpha > 0$. When $\alpha < 0$, $\dot{R} > 0$ and the theorem may be established by using equation (5.3), while (5.2) can be used to determine the solution. The same remark will apply to almost all the examples considered here.

The problem of solving the equation for $R(t)$ will be considered later. Here we note that a good approximation is obtained by setting $\varrho \equiv 0$ in the right hand side of the equation. Then

$$\dot{R} \sim -\frac{\alpha}{\sqrt{\pi t}} (1 - \exp\{-(M/\sqrt{t})^2\}),$$

$$R \sim -2\frac{\alpha}{\sqrt{\pi}} [\sqrt{t}(1 - \exp\{-(M/\sqrt{t})^2\} + 2M \int_{M/\sqrt{t}}^{\infty} \exp\{-\sigma^2\}\, d\sigma].$$

3. *Decay by evaporation (or growth by condensation) of a liquid drop.* This is the three-dimensional analogue of example 1. Assuming that in the process of evaporation (or condensation) the drop will remain spherical due to some extraneous mechanism like surface tension, and that the saturation density of the vapor is independent of the radius, the radius of the drop $R(t)$, and the density of the surrounding vapor $u(r, t)$ will be determined by

DE	$\Delta u = u_t$	for $r > R(t)$, $t > 0$,
BC	$u(r, 0) = 0$	for $r > R(0) = 1$,
FBC1	$u(r, t) = 1$	for $t \geqq 0$,
FBC2	$\alpha u_r(R, t) = \dot{R}$	for $t > 0$.

The derivation of these equations is similar to that of example 1, and they have been written in dimensionless form, using for the unit of length the initial radius of the drop, l.

As is well known, $c = ru(r, t)$ satisfies the one-dimensional heat equation. Using c, the problem is now formulated as follows:

DE	$c_{rr} = c_t$	for $r > R(t)$, $t > 0$,
BC	$c(r, 0) = 0$	for $r > R(0) = 1$,
FBC1	$c(R, t) = R$	for $t \geqq 0$,
FBC2	$\alpha u_r(R, t) = R\dot{R} + \alpha$	for $t > 0$.

In case of condensation $(-1 < \alpha < 0)$ the drop will grow ad infinitum. In case $\alpha > 0$, it is expected that the drop will disappear in a finite time t_0, i.e., $R(t_0) = 0$. For $t > t_0$ there will of course be no boundary to determine, and the free boundary conditions will be replaced by

$$u_r(0, t) = 0, \qquad \text{for } t > t_0,$$

or

$$\lim_{r \to 0} r^{-1} c(r, t) = \text{finite}, \qquad \text{for } t > t_0.$$

Here, $f[\varrho] = \varrho$, $g[\varrho] = \dfrac{\varrho\dot{\varrho}}{\alpha} + 1$, $D_\varrho^+ = E_\varrho^+$, $a = 1$, and

$$\Gamma(u) = \begin{Bmatrix} u(x, 0) \\ u(\infty, t) \end{Bmatrix} = 0 \qquad \begin{matrix} x > 1 \\ t \geqq 0. \end{matrix}$$

The complementary part is zero, while the fundamental part is

$$\begin{aligned} v^\varrho = u^\varrho &= \tfrac{1}{2} \int_0^t \left(\frac{\varrho(\tau)}{t - \tau} z(x, \tau) - \frac{\alpha^{-1}\varrho(\tau)\dot{\varrho}(\tau) + 1 + \varrho(\tau)\dot{\varrho}(\tau)}{\sqrt{t - \tau}} \right) \eta(z(x, \tau))\, d\tau \\ &= -\sqrt{t}\, \eta(z(x, 0)) + x \int_{z(x,0)}^{z(x,t)} \eta(\sigma) d\sigma - \frac{1}{2\alpha} \int_0^t \frac{\varrho(\tau)\dot{\varrho}(\tau)}{\sqrt{t - \tau}} \eta(z(x, \tau))\, d\tau. \end{aligned}$$

For the equation for the boundary one now obtains

$$\text{(A)} \quad \varrho\dot{\varrho} = -2\alpha \int_{-\infty}^{z(\varrho,0)} \eta(\sigma) d\sigma - \frac{\alpha}{\sqrt{t}} \eta(z(\varrho, 0)) + \int_0^t \frac{\varrho(\tau)\dot{\varrho}(\tau)}{t - \tau} z(\varrho, \tau) \eta(z(\varrho, \tau))\, d\tau,$$
$$\varrho(0) = 1.$$

When studying a free boundary problem one generally feels that an extension of the solution into regions where the final result is of no interest will only complicate the problem.[5] In this example, stemming from a three-dimensional problem, one almost fears extending our considerations into the region of negative r. Let us then see what our procedure would lead to had we not made this extension.

[5]To our knowledge, the only previous exception to this attitude is found in [15], where analytic continuation is used to produce solutions to the Helmholtz problem.

We restrict the domain under consideration to the point set $t > 0$, $r > 0$. The domain D_ϱ^- is therefore the point set $(t > 0, 0 < x < \varrho(t))$, and at $x = 0$ we impose the boundary condition,

$$u^\varrho(0, t) = 0.$$

The auxiliary functional $\bar{u}^\varrho$ is easily found by using the method of images. If $v^\varrho(x, t)$ is the fundamental part—the same as the one obtained previously—,

$$\bar{u}^\varrho(x, t) = v^\varrho(x, t) - v^\varrho(-x, t), \qquad x \geqq 0.$$

Since $\varrho(t) > 0$ for $t < t_0$, $v^\varrho(-x, t)$ is a regular solution of the heat equation in D.

The equation for the boundary is again

$$\bar{u}_x^{\varrho-}(\varrho(t), t) = 0.$$

Now, in view of the jump conditions,

$$v_x^{\varrho-}(\varrho(t), t) = v_x^\varrho(\varrho(t), t) - \tfrac{1}{2}(\alpha^{-1}\varrho\dot{\varrho} + 1),$$

while

$$v_x^{\varrho-}(-\varrho(t), t) = v_x^\varrho(-\varrho(t), t).$$

The equation for the boundary is therefore

$$\text{(B)} \qquad \varrho\dot{\varrho}(t) = 2\alpha[v_x^\varrho(\varrho(t), t) - \tfrac{1}{2} + v_x^\varrho(-\varrho(t), t)], \qquad \varrho(0) = 1.$$

Equation (B) differs from the equation (A) by an additional term, $2\alpha v_x^\varrho(-\varrho(t), t)$ and thus appears more complicated. We will show that every solution $R(t)$ of (A) also solves (B). Indeed, if $R(t)$ is a solution of (A), then $v^{R-}(x, t) \equiv 0$, and consequently $v_x^{R-}(-R(t), t) = v_x^R(-R(t), t) \equiv 0$. Hence for $\varrho(t) = R(t)$—and in this case only—the left hand sides of both equations coincide. Furthermore, since for $x > R(t)$, $-x < -R(t) < R(t)$, $v^{R+}(-x, t) \equiv 0$ in D_R^+. Hence,

$$u^{R+} \equiv \bar{u}^{R+}, \qquad P \in D_R^+,$$

although in general, $u^\varrho \not\equiv \bar{u}^\varrho$. It would be quite difficult to verify the converse had we not known it in advance.

4. *A theory of cloud behavior.*

Let the cloud consist of a collection (finite or denumerably infinite) of disjoint spherical drops growing or decaying according to the mechanism assumed in example 3. Let $\boldsymbol{\xi}_i$ be the center of the i-th drop, $R_i(t)$ its radius, $R_i(0) = l_i$. Denote by $D_i(t)$ the point set $|\boldsymbol{x} - \boldsymbol{\xi}_i| > R_i(t)$, and by S_i its boundary $|\mathbf{x} - \boldsymbol{\xi}_i| = R_i(t)$. The problem is to determine the density function $u(\boldsymbol{x}, t)$ and the radia $R_i(t)$ satisfying:

$$\begin{array}{llll}
\text{DE} & \Delta u = u_t & \text{for } \mathbf{x} \in \bigcap_i D_i(t), & t > 0, \\
\text{BC} & u = 0 & \text{for } \mathbf{x} \in \bigcap_i D_i(0), & \\
\text{FBC1}_i & u = 1 & \text{for } \mathbf{x} \in S_i,\ t \geqq 0, & i = 1, 2, \ldots, \\
\text{FBC2}_i & \dfrac{\alpha}{4\pi} \displaystyle\iint_{\mathbf{x} \in S_i} u_r(x, t)\, d\omega = \dot{R}_i(t) & \text{for } t > 0, & i = 1, 2, \ldots .
\end{array}$$

In this formulation, the problem of cloud behavior is of much greater complexity than the preceeding problems in that the concentration satisfies a three-dimensional heat equation, and in that there are ***many*** free boundaries to determine. While the method devised in this Part 1 is inadequate to treat this problem, it will be dealt with later when extensions of our method are considered.

For some purposes the behavior of the cloud is satisfactorily described by assuming that all drops are initially of the same radius and that their centers form a cubic lattice. The problem is then identical with that of a single spherical drop located at the center of a cube whose boundary is impervious to vapor. If the side of the cube is large compared to the radius of the drop, a further satisfactory approximation is obtained on assuming that the cube is replaced by a concentric sphere of the same volume. The density function u then depends only on the distance from the center of the drop, and the problem becomes identical with Problem 3, except that

$$\begin{array}{ll}
u(x, t) \text{ is defined for } R(t) < r < M, & t > 0, \\
u_r(M, t) = 0. &
\end{array}$$

Here M is the dimensionless radius of the sphere. Using again the function $c = ru(r, t)$ the problem is formulated as follows:

$$\begin{array}{lll}
\text{DE} & c_{rr} = c_t & \text{for } R(t) < r < M,\ t > 0, \\
\text{BC} & \begin{cases} c(r, 0) = 0 \\ Mc_r(M, t) - c(M, t) = 0 \end{cases} & \begin{array}{l} \text{for } 1 = R(0) < r < M, \\ \text{for } t \geqq 0, \end{array} \\
\text{FBC1} & c(R, t) = R & \text{for } t \geqq 0, \\
\text{FBC2} & \alpha c_r(R, t) = R\dot{R} + \alpha & \text{for } t > 0.
\end{array}$$

As in example 3, $f[\varrho] = \varrho$, $g[\varrho] = \dfrac{\varrho\dot{\varrho}}{\alpha} + 1$, but $D_\varrho^+[\varrho(t) < x < M,$ $t > 0]$, $a = 1$, and

$$\Gamma u = \begin{Bmatrix} u(x, 0) \\ Mu_x(M, t) - u(M, t) \end{Bmatrix} = 0 \qquad \begin{array}{l} \text{for } 1 < x < M \\ \text{for } t \geqq 0. \end{array}$$

The fundamental part v is the same as in Problem 3, and the complementary w part satisfies,

DE	$Lw^{\varrho} = 0,$	$P \in D,$
IC	$w^{\varrho}(x, 0) = 0,$	$x \leqq M,$
BC	$Mw_x^{\varrho}(M,t) - w^{\varrho}(M,t) = -(Mv_x^{\varrho}(M,t) - v^{\varrho}(M,t)),$	$t \geqq 0$
∞C	$\lvert w^{\varrho}(x, t)\rvert \leqq A, \; \lvert\sqrt{t}\, w_x^{\varrho}(x, t)\rvert \leqq A.$	

One finds,

$$w^{\varrho}(x,t) = \sqrt{t}\eta(z(2M-x,0)) + \int_{z(2M-x,0)}^{\infty} \eta(\sigma)\, d\sigma - \frac{1}{2\alpha}\int_0^t \frac{\varrho(\tau)\dot{\varrho}(\tau)}{\sqrt{t-\tau}}$$

$$\eta(z(2M-x,\tau))\, d\tau - 2(M-1)\exp\left\{\frac{t}{M^2} - \frac{2M-x-1}{M}\right\}\int_{-\frac{\sqrt{t}}{M}+z(2M-x,0)}^{\infty} \eta(\sigma)d\sigma$$

$$-\frac{2}{\alpha M}\int_0^t \varrho(\tau)\dot{\varrho}(\tau)\exp\left\{\frac{t-\tau}{M^2} - \frac{2M-x-\varrho(\tau)}{M}\right\}\left(\int_{-\frac{\sqrt{t-\tau}}{M}+z(2M-x,\tau)}^{\infty} \eta(\sigma)d\sigma\right) d\tau.$$

The simplest equation that the boundary function must satisfy appears to be the equation

$$Mu_x^{\varrho-}(\varrho, t) - u^{\varrho-}(\varrho, t) = 0,$$

or, explicitly,

$$\begin{aligned}\varrho\varrho(t) = &-2\alpha\left(1 - \frac{\varrho(t)}{M}\right)\left(\int_{-\infty}^{z(\varrho,0)} \eta(\sigma)d\sigma - \int_{z(2M-\varrho,0)}^{\infty} \eta(\sigma)d\sigma\right)\\ &- \frac{\alpha}{\sqrt{t}}\left(1 - \frac{2t}{M}\right)\left(\eta(z(\varrho, 0)) - \eta(z(2M-\varrho, 0))\right)\\ &+ \int_0^t \frac{\varrho(\tau)\dot{\varrho}(\tau)}{t-\tau}\left[\left(z(\varrho,\tau) + \frac{\sqrt{t-\tau}}{M}\right)\eta(z(\varrho,\tau))\right.\\ &\left. - \left(z(2M-\varrho,\tau) + \frac{\sqrt{t-\tau}}{M}\right)\eta(z(2M-\varrho,\tau))\right] d\tau,\end{aligned}$$

$\varrho(0) = 1.$

See [12].

5. *Dissolution of gas bubble in liquid.*[6]

The radius $R(t)$ of a gas bubble submerged in a liquid, and the concentration $u(r, t)$ of the gas dissolved in the liquid satisfy the following equations with $c(r, t) = ru(r, t)$:

DE	$c_{rr} = c_t$	for $r > R(t),\ t > 0,$
BC	$c(r, 0) = 0$	for $r > R(0) = 1,$
FBC1	$c(R, t) = (1-l)R + l$	for $t \geqq 0,$
FBC2	$\alpha c_r(R, t) = (R+m)\dot{R} + \alpha\left(1 - l + \frac{l}{R}\right)$	for $t > 0.$

[6] A simplified formulation according to J. B. Keller.

Here, l, m, α are constants, and $l \geqq 0, m \geqq 0, \alpha > -1$. In this formulation the effects of surface tension are included. When these effects are neglected, $l = m = 0$, and the problem becomes identical with Problem 3.

$$\text{Here, } f[\varrho] = (1-l)\varrho + l,\ g[\varrho] = \frac{(\varrho + m)\dot{\varrho}}{\alpha} + 1 - l + \frac{l}{\varrho},\ D_\varrho^+ = E_\varrho^+,$$

$a = 1$, and

$$\Gamma u = \begin{Bmatrix} u(x, 0) \\ u(\infty, t) \end{Bmatrix} = 0 \qquad \begin{matrix} \text{for } x > 1 \\ \text{for } t \geqq 0. \end{matrix}$$

This problem is similar to Problem 3. The complementary part is zero and one finds for the fundamental part of the auxiliary functional

$$v^\varrho = u^\varrho = -(1-l)\sqrt{t}\,\eta(z(x,0)) + [(1-l)x + l] \int_{z(x,0)}^{z(x,t)} \eta(\sigma) d\sigma$$
$$- \frac{1}{2\alpha} \int_0^t \frac{(\varrho(\tau) + m)\dot{\varrho}(\tau) + \alpha l \varrho^{-1}(\tau)}{\sqrt{t-\tau}} \eta(z(x,\tau))\, d\tau.$$

Equation (5.2) for the boundary is then

$$(\varrho(t) + m)\dot{\varrho}(t) = -2\alpha(1-l) \int_{-\infty}^{z(\varrho,0)} \eta(\sigma)\, d\sigma - \frac{\alpha}{\sqrt{t}} \eta(z(\varrho, 0)) - \alpha l \varrho^{-1}(t)$$
$$+ \int_0^t \frac{(\varrho(\tau) + m)\dot{\varrho}(\tau) + \alpha l \varrho^{-1}(\tau)}{t - \tau} z(\varrho, \tau) \eta(z(\varrho, \tau))\, d\tau,$$

$\varrho(0) = 1$.

6. *Change of phase without appreciable change of density (e.g., recrystallization).*

When heat is supplied to a solid at a recrystallization temperature, a partial recrystallization will occur. Part of the heat will be used to supply the latent heat of recrystallization while the remainder is used to increase the

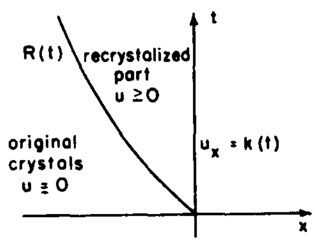

Figure 5

temperature of the recrystallized solid. The moving front, $R(t)$ and the temperature $u(x, t)$ of the recrystallized part are determined in appropriate dimensionless variables by solving

$$
\begin{array}{lll}
\text{DE} & u_{xx} = u_t & \text{for } 0 < x < R(t),\ t > 0, \\
\text{BC} & \begin{cases} u_x(0, t) = k(t), \\ R(0) = 0, \end{cases} & \\
\text{FBC1} & u(R, t) = 0 & \text{for } \quad t \geqq 0, \\
\text{FBC2} & u_x(0, t) = -\dot{R} & \text{for } \quad t > 0.
\end{array}
$$

Here $k(t) \geqq 0$ is the heat input function (in appropriate units), and it has been assumed that the recrystallization temperature is normalized to zero. Also $f[\varrho] = 0$, $g[\varrho] = -\dot{\varrho}$, $D_\varrho^+[\varrho(t) < x < 0]$, $a = 0$, and

$$\Gamma u = u_x(0, t) - k(t) = 0, \qquad t \geqq 0.$$

The fundamental part is

$$v^\varrho = \tfrac{1}{2} \int_0^t \frac{\dot{\varrho}(\tau)}{\sqrt{t-\tau}} \eta(z(x, \tau))\, d\tau.$$

The complementary part is obtained by using the method of images. One finds

$$w^\varrho = \tfrac{1}{2} \int_0^t \frac{\dot{\varrho}(\tau)}{\sqrt{t-\tau}} \eta(z(-x, \tau))\, d\tau + \int_0^t \frac{k(\tau)}{\sqrt{t-\tau}} \eta\left(\frac{x}{2\sqrt{t-\tau}}\right) d\tau.$$

The equation (5.2) for the boundary is

$$\dot{\varrho}(t) = \varrho(t) \int_0^t \frac{k(\tau)}{(t-\tau)^{3/2}} \eta\left(\frac{\varrho(t)}{2\sqrt{t-\tau}}\right) d\tau + \int_0^t \frac{\dot{\varrho}(\tau)}{t-\tau} [z(\varrho, \tau)\eta(z(\varrho, \tau)) - z(-\varrho, \tau)\eta(z(-\varrho, \tau))]\, d\tau,$$

$\varrho(0) = 0.$

7. Generalizations. Problems of Freezing of a Lake and of Solidification of the Terrestrial Crust

So far we considered the simplest type of free boundary problem for the heat equation. Two kinds of complications could be conceived. First there may be more than one free boundary. Secondly, the differential equation may be different from the one considered. To see under what conditions our method can be applied to more complicated cases, let us briefly review the main factors which contributed to the establishment of the reduction theorems of Section 5.

We associated with the free boundary an auxiliary functional having appropriate jumps across the candidates for the free boundary. The construction of the fundamental part of the auxiliary functional depended essentially on the existence of a fundamental solution of the equation in the

large. Finally, the reduction theorem could be established mainly because of the uniqueness theorem for the first boundary value problem for the heat equation.

Disregarding then all technical difficulties, an auxiliary functional could be constructed whenever the differential equation has a fundamental solution in the large. If there are more than one free boundary, this functional will contain in addition to the complementary part the sum of as many fundamental parts as there are free boundaries. Each of these will have appropriate jumps across the corresponding boundary, being regular elsewhere. To establish the reduction theorem we would need again the uniqueness property of solutions for the first boundary value problem.

Clearly then our procedure can be extended to problems involving the heat equation in more than two independent variables, and to free boundary problems for the potential equation. The latter lead to certain complications which are now being studied.

Here we shall consider still another kind of extension. Namely, we shall consider problems involving *two media*, each governed by a *different* one-dimensional heat equation and separated by a free boundary. Since no additional preparation is required we proceed immediately to application to two still unsolved problems of great historical importance.

1. *Freezing of a lake of finite depth.*

We assume that the lake has a constant depth h, and that its bottom and its lateral boundaries are nonconducting. Let

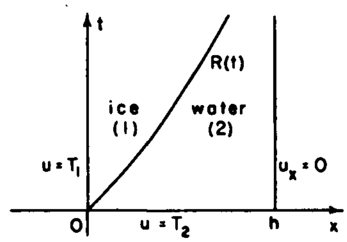

Figure 6

$c_{1,2}$ specific heat
$d_{1,2}$ density
$k_{1,2}$ coefficient of conductivity
L latent heat of fusion

$$\sqrt{\frac{k_1}{c_1 d_1}} = a_1 , \qquad \sqrt{\frac{k_2}{c_2 d_2}} = a_2 .$$

Subscript 1 will refer to conditions in the layer of ice, subscript 2 to conditions in water. We shall assume that in the process of freezing there is no

heat transfer by convection, and—what is more disturbing—that all the parameters remain fixed, and that in particular $d_1 = d_2 = d$. Then the temperature $u(x, t)$ and the freezing front $R(t)$ satisfy:

$$\begin{array}{lll} \text{DE} & a_1^2 u_{1,xx} = u_{1,t}, & \left.\begin{array}{r} 0 < x < R(t) \\ t \geq 0 \end{array}\right\} \text{in ice,} \\ \text{BC} & u_1(0, t) = T_1, & \end{array}$$

$$\begin{array}{lll} \text{DE} & a_2^2 u_{2,xx} = u_{2,t}, & \left.\begin{array}{r} R(t) < x < h \\ 0 = R(0) < x < h \\ t \geq 0 \end{array}\right\} \text{in water,} \\ \text{BC} & \left\{\begin{array}{l} u_2(x, 0) = T_2, \\ u_{2,x}(h, t) = 0, \end{array}\right. & \end{array}$$

$$\begin{array}{ll} \text{FBC1} & u_1(R(t), t) = u_2(R(t), t) = 0, \\ \text{FBC2} & k_1 u_{1,x}(R(t), t) - k_2 u_{2,x}(R(t), t) = Ld\dot{R}(t) \end{array} \Bigg\} \begin{array}{l}\text{at the} \\ \text{freezing front.}\end{array}$$

Let us first consider the problems in ice and water as two separate problems. Let

$$u_{1,x}(R(t), t) = g_1(t),$$
$$u_{2,x}(R(t), t) = g_2(t).$$

The auxiliary functional u_1^ϱ for the problem in ice is

$$u_1^\varrho(x, t) = \frac{a_1}{2}\int_0^t \frac{g_1(\tau)}{\sqrt{t-\tau}}[\eta(z_1(x, \tau)) - \eta(z_1(-x, \tau))]\, d\tau + 2T_1\int_{\frac{x}{2a_1\sqrt{t}}}^{\infty} \eta(\sigma)\, d\sigma,$$

$$z_1(x, \tau) = \frac{x - \varrho(\tau)}{2a_1\sqrt{t-\tau}}, \qquad P \in D_{1,\varrho}^- \cup D_{1,\varrho}^+,$$

where $D_{1,\varrho}^-$ is the point set $(t > 0,\ 0 < x < \varrho(t))$ and $D_{1,\varrho}^+$ is the point set $(t > 0,\ x > \varrho(t))$. For the problem in water we get

$$u_2^\varrho(x, t) = -\frac{a_2}{2}\int_0^t \frac{g_2(\tau)}{\sqrt{t-\tau}}[\eta(z_2(x, \tau)) + \eta(z_2(2h-x, \tau))\, d\tau + T_2\int_{(x-2h)/2a_2\sqrt{t}}^{x/(2a_2\sqrt{t})} \eta(\sigma) d\sigma,$$

$$z_2(x, \tau) = \frac{x - \varrho(\tau)}{2a_2\sqrt{t-\tau}}, \qquad P \in D_{2,\varrho}^- \cup D_{2,\varrho}^+,$$

where $D_{2,\varrho}^-$ is the point set $(t > 0,\ x < \varrho(t))$, and $D_{2,\varrho}^+$ is the point set $(t > 0,$ $\varrho(t) < x < h)$. The equations that the boundary curve must satisfy are: for the problem in ice, $u_{1,x}^{\varrho+}(\varrho, t) = 0$, or $g_1(t) = 2u_{1,x}^\varrho(\varrho, t)$, $\varrho(0) = 0$, for the problem in water, $u_{2,x}^{\varrho-}(\varrho, t) = 0$, or $g_2(t) = 2u_{2,x}(\varrho, t)$, $\varrho(0) = 0$.

We now recall that these are not independent problems, but that $g_1(t)$ and $g_2(t)$ are related by FBC2. Thus we get three equations for the three unknowns, $g_1(t)$, $g_2(t)$, and $\varrho(t)$. These equations are, explicitly,

$$g_1(t) = -\int_0^t \frac{g_1(\tau)}{t-\tau}\left[z_1(\varrho,\tau)\eta(z_1(\varrho,\tau)) + z_1(-\varrho,\tau)\eta(z_1(-\varrho,\tau))\right]d\tau - \frac{2T_1}{a_1\sqrt{t}}\eta(z_1(\varrho,0)),$$

$$g_2(t) = \int_0^t \frac{g_2(\tau)}{t-\tau}\left[z_2(\varrho,\tau)\eta(z_2(\varrho,\tau)) - z_2(2h-\varrho,\tau)\eta(z_2(2h-\varrho,\tau))\right]d\tau + \frac{T_2}{a_2\sqrt{t}}\left[\eta(z_2(\varrho,0)) - \eta(z_2(2h-\varrho,0))\right],$$

$$\dot{\varrho}(t) = \frac{k_1}{dL}g_1(t) - \frac{k_2}{dL}g_2(t), \qquad \varrho(0) = 0.$$

2. *Solidification of the terrestrial crust.*

This is the three-dimensional analogue of the preceeding problem. Subscript 1 refers to the crust, subscript 2 to the liquid interior of the earth, all symbols having a meaning similar to that in the preceeding problem. The temperature $u(r, t)$ and the solidification front $R(t)$ are determined by

$$\begin{array}{lll}
\text{DE} & a_1^2\Delta u_1 = u_{1,t}, & \left.\begin{array}{r} R(t) < r < a \\ t \geqq 0 \end{array}\right\} \begin{array}{l}\text{in the crust}\\ a = \text{radius of the earth,}\end{array} \\
\text{BC} & u_1(a, t) = T_1, & \\
\text{DE} & a_2^2\Delta u_2 = u_{2,t}, & \left.\begin{array}{r} 0 < r < R(t) \\ 0 < r < R(0) = a \\ t \geqq 0 \end{array}\right\} \text{in the liquid,} \\
\text{BC} & \left\{\begin{array}{l} u_2(r, 0) = T_2, \\ u_{2,r}(0, t) = 0, \end{array}\right. & \\
\text{FBC1} & u_1(R, t) = u_2(R, t) = 0 & \left.\right\} \begin{array}{l}\text{at the}\\ \text{solidification front.}\end{array} \\
\text{FBC2} & k_1u_{1,x}(R, t) - k_2u_{2,r}(R, t) = Ld\dot{R}(t) &
\end{array}$$

As in examples 3—5 of Section 6 it is again convenient to introduce $c = ru(r, t)$. $c(r, t)$ then satisfies:

$$\begin{array}{lll}
\text{DE} & a_1^2c_{1,rr} = c_t, & \left.\begin{array}{r} R(t) < r < a \\ t \geqq 0 \end{array}\right\} \text{in the crust,} \\
\text{BC} & c_1(a, T) = T_1a, & \\
\text{DE} & a_2^2c_{2,rr} = c_{2,t}, & \left.\begin{array}{r} 0 < r < R(t) \\ 0 < r < R(0) = a \\ t \geqq 0 \end{array}\right\} \text{in the liquid.} \\
\text{BC} & \left\{\begin{array}{l} c_2(r, 0) = T_2r, \\ c_2(0, t) = 0, \end{array}\right. & \\
\text{FBC1} & c_1(R(t), t) = c_2(R(t), t) = 0, & \\
\text{FBC2} & k_1c_{1,r}(R(t), t) - k_2c_{2,r}(R(t), t) = LdR(t)\dot{R}(t). &
\end{array}$$

Proceeding as in the previous example, we find for the auxiliary functionals:

$$u_1^\varrho(x, t) = -\frac{a_1}{2}\int_0^t \frac{g_1(\tau)}{\sqrt{t-\tau}}\left[\eta(z_1(x,\tau)) - \eta(z_1(2a-x,\tau))\right] + aT_1\int_{-\infty}^{(a-x)/2a_1\sqrt{t}} \eta(\sigma)d\sigma, \quad P \in D_{1,\varrho}^- \cup D_{1,\varrho}^+,$$

$$u_2^0(x,t)=\frac{a_2}{2}\int_0^t\frac{g_2(\tau)}{\sqrt{t-\tau}}\left[\eta(z_2(x,\tau))-\eta(z_2(-x,\tau))\right]d\tau+T_2x\int_{(x-a)/2a_2\sqrt{t}}^{(x+a)/2a_2\sqrt{t}}\eta(\sigma)d\sigma$$
$$+T_2a_2\sqrt{t}\left[-\eta\left(\frac{x-a}{2a_2\sqrt{t}}\right)+\eta\left(\frac{x+a}{2a_2\sqrt{t}}\right)\right],\qquad P\in D_{2,\varrho}^-\cup D_{2,\varrho}^+.$$

Here the various point sets are defined as follows:

$$D_{1,\varrho}^-\ [t>0,\ -\infty<x<\varrho(t)],$$
$$D_{1,\varrho}^+\ [t>0,\ \varrho(t)<x<a],$$
$$D_{2,\varrho}^-\ [t>0,\ 0<x<\varrho(t)],$$
$$D_{2,\varrho}^+\ [t>0,\ \varrho(t)<x<+\infty].$$

To determine the position of the interface, one has to solve a system of three equations for $g_1(t)$, $g_2(t)$ and $\varrho(t)$, namely,

$$g_1(t)=\int_0^t\frac{g_1(\tau)}{t-\tau}\left[z_1(\varrho,\tau)\eta(z_1(\varrho,\tau))+z_1(2a-\varrho,\tau)\eta(z_1(2a-\varrho,\tau))\right]d\tau$$
$$-\frac{T_2a}{a_1\sqrt{t}}\eta(z_1(\varrho,0)),$$

$$g_2(t)=-\int_0^t\frac{g_2(\tau)}{t-\tau}\left[z_2(\varrho,\tau)\eta(z_2(\varrho,\tau))+z_2(-\varrho,\tau)\eta(z_2(-\varrho,\tau))\right]d\tau$$
$$+2T_2\int_{z_2(\varrho,0)}^{z_2(\varrho+2a,0)}\eta(\sigma)d\sigma-\frac{T_2a}{a_2\sqrt{t}}\left[\eta(z_2(\varrho,0))+\eta(z_2(\varrho+2a,0))\right],$$

$$\varrho\dot{\varrho}(t)=\frac{k_1}{dL}g_1(t)-\frac{k_2}{dL}g_2(t),\qquad \varrho(0)=a.$$

In connection with the last two examples we note that Rubinstein [14], obtained in a similar problem the same equations for the free boundary that we would get by applying the method of this paper. The problem considered by Rubinstein is a generalization of problems of Stefan (see [1]), namely to find $u_1(x,t)$, $u_2(x,t)$, and $R(t)$ satisfying:

DE	$a_1^2u_{1,xx}=u_{1,t}$,	$t>0,\ x<R(t)$,
IC	$u_1(x,0)=\varphi_1(x)$,	$x<0$,
DE	$a_2^2u_{2,xx}=u_{2,t}$,	$t>0,\ x>R(t)$,
IC	$u_2(x,0)=\varphi_2(x)$,	$x>0$,
FBC1	$u_1(R(t),t)=u_2(R(t),t)=0$,	
FBC2	$k_1u_{1,x}(R(t),t)-k_2u_{2,x}(R(t),t)=k\dot{R}(t)$.	

(When $\varphi_1(x)$ and $\varphi_2(x)$ are both constants, a solution can be obtained in closed form.)

It was first observed that if a solution to the problem exists, then u_1 and u_2 have the following representations derived from Green's formula:

$$u_1(x,t) = \frac{a_1}{2}\int_0^t \frac{g_1(\tau)}{\sqrt{t-\tau}}\eta(z_1(x,\tau))\,d\tau + \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^0 \varphi_1(\xi)\exp\{-[(x-\xi)/2a_1\sqrt{t}]^2\}\,d\xi, \quad x < R(t),$$

$$u_2(x,t) = -\frac{a_2}{2}\int_0^t \frac{g_2(\tau)}{\sqrt{t-\tau}}\eta(z_2(x,\tau))\,d\tau + \frac{1}{2\sqrt{\pi t}}\int_0^\infty \varphi_2(\xi)\exp\{-[(x-\xi)/2a_2\sqrt{t}]^2\}\,d\xi, \quad x > R(t),$$

where $g_1(t) = u_{1,x}(R(t), t)$, $g_2(t) = u_{2,x}(R(t), t)$. The equations which the boundary function must satisfy were then derived by requiring that

$$\lim_{R(t)>x\to R(t)} u_{1,x}(x,t) = g_1(t),$$
$$\lim_{R(t)<x\to R(t)} u_{2,x}(x,t) = g_2(t),$$

and adjoining the FBC2. We note that this attack would prove unsuccessful if the domain of definition of u_1 or u_2 were finite.

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Received July, 1955.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. IX, 33–43 (1956)

On a Class of Solutions of Maxwell's Electromagnetic Equations

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§1. In discussions of electromagnetic propagation in a homogeneous isotropic conducting medium [1], it is frequently found convenient to regard the solutions as built up from the real parts of component plane wave solutions having the "damped monochromatic" form

$$\mathbf{E} = \mathbf{A} \exp \{i(\omega t - k\mathbf{n} \cdot \mathbf{x}) - \alpha_0 t - \beta_0 \mathbf{n} \cdot \mathbf{x}\},$$
$$\mathbf{H} = \mathbf{B} \exp \{i(\omega t - k\mathbf{n} \cdot \mathbf{x} - \psi) - \alpha_0 t - \beta_0 \mathbf{n} \cdot \mathbf{x}\}.$$

Here $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic vectors respectively, $\mathbf{A}$ and $\mathbf{B}$ are thus real constant vector amplitudes, and $\mathbf{n}$ is a unit vector in the direction of propagation; k and β_0 are the propagation and absorption constants, ω and α_0 are the frequency and decay constants, and ψ is a constant phase difference. $\mathbf{A}$ and $\mathbf{B}$ may be shown to be orthogonal.

The form of such solutions suggests that one might obtain more general ones by substituting a real general phase function $\varphi(x, y, z)$ for the linear function $-k\mathbf{n} \cdot \mathbf{x}$ and real general vector amplitude functions $\mathbf{E}_0(x, y, z)$ and $\mathbf{H}_0(x, y, z)$ for the factors $\mathbf{A} \exp \{-\beta_0 \mathbf{n} \cdot \mathbf{x}\}$ and $\mathbf{B} \exp \{-\beta_0 \mathbf{n} \cdot \mathbf{x}\}$, respectively. The quantity ψ, too, might be allowed to depend upon position. If such substitutions are made, the resulting expressions represent "waves" which are in general no longer plane, and do not have constant amplitudes. It is clear however that Maxwell's electromagnetic equations will impose certain restrictions on φ, ψ, $\mathbf{E}_0$ and $\mathbf{H}_0$, which it is the purpose of this paper to investigate.

§2. Maxwell's equations for a homogeneous isotropic conducting medium are

$$\begin{aligned} &\nabla \times \mathbf{E} + \mu \frac{\partial \mathbf{H}}{\partial t} = 0, \\ (1) \qquad &\nabla \times \mathbf{H} - \varepsilon \frac{\partial \mathbf{E}}{\partial t} = \sigma \mathbf{E}, \\ &\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{H} = 0, \end{aligned}$$

*Work performed chiefly while Professor Hudson was in residence at the Navy Electronics Laboratory, summer of 1953.

μ, ε, and σ being the permeability, dielectric constant, and conductivity, respectively, of the medium. We look for electric and magnetic vectors $\mathbf{E}$ and $\mathbf{H}$ satisfying (1) of the form

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 \exp\{-\alpha_0 t + i(\varphi + \omega t)\}, \\ \mathbf{H} &= \mathbf{H}_0 \exp\{-\alpha_0 t + i(\varphi - \psi + \omega t)\}, \end{aligned} \tag{2}$$

where $\mathbf{E}_0$, $\mathbf{H}_0$, φ and ψ are real functions of position and α_0 and ω are real constants.

§3. Certain preliminary results will be needed.

LEMMA 1. [2] *A necessary and sufficient condition that a vector field* $\mathbf{V}$ *be of the form* $\mathbf{V} = f\nabla h$, *where* f *and* h *are scalar functions of position, is that* $\mathbf{V}\cdot\nabla\times\mathbf{V} = 0$.

LEMMA 2. *If* (i) $|\nabla\mathscr{I}|^2 = 1$,
(ii) $\nabla\mathscr{E}\cdot\nabla\mathscr{H} = \nabla\mathscr{E}\cdot\nabla\mathscr{I} = \nabla\mathscr{H}\cdot\nabla\mathscr{I} = 0$,
(iii) $|\nabla\mathscr{E}| = |\nabla\mathscr{H}|\,g(\mathscr{I})$,

where $\mathscr{E}, \mathscr{H}, \mathscr{I}$ *are independent scalar functions of position, and* g *is a non-vanishing function of* $\mathscr{I}$ *only,*

$$\textit{then } \nabla^2\mathscr{E} = \nabla^2\mathscr{H} = 0.$$

Proof: We first note that $\nabla\mathscr{I} = \pm\dfrac{\nabla\mathscr{E}\times\nabla\mathscr{H}}{|\nabla\mathscr{E}|\,|\nabla\mathscr{H}|}$.

Then we consider $\mathscr{I}\nabla\mathscr{E}$. We have

$$\nabla\times(\mathscr{I}\nabla\mathscr{E}) = \nabla\mathscr{I}\times\nabla\mathscr{E} = \pm\, g\nabla\mathscr{H}.$$

Thus
$$\begin{aligned} \nabla\cdot(g\nabla\mathscr{H}) &= g'\nabla\mathscr{I}\cdot\nabla\mathscr{H} + g\nabla^2\mathscr{H} \\ &= g\nabla^2\mathscr{H} \\ &= \pm\,\nabla\cdot\nabla\times(\mathscr{I}\nabla\mathscr{E}) \\ &= 0. \end{aligned}$$

Hence $\nabla^2\mathscr{H} = 0$. Similarly $\nabla^2\mathscr{E} = 0$.

LEMMA 3. *Necessary and sufficient conditions that there exist functions* $\mathscr{E}, \mathscr{H}, \mathscr{I}$ and $g(\mathscr{I})$, *satisfying the conditions of Lemma 2 are that either:*
(i) $g(\mathscr{I}) = $ *constant, and*
(ii) *the line element in Euclidean space must be expressible in the form*

$$ds^2 = p^2(\mathscr{E},\mathscr{H})(m\mathscr{I} + n)^2(d\mathscr{E}^2 + d\mathscr{H}^2) + d\mathscr{I}^2 \tag{3}$$

where

$$\left(\frac{\partial^2}{\partial\mathscr{E}^2} + \frac{\partial^2}{\partial\mathscr{H}^2}\right)\ln p + m^2p^2 = 0 \tag{4}$$

and m and n are constants, or: (i)′ g *or* g^{-1} *is a linear function of* $\mathscr{I}$, *and* (ii)′ *the line element is expressible either in the form*

$$ds^2 = d\mathscr{E}^2 + \mathscr{I}^2 d\mathscr{H}^2 + d\mathscr{I}^2 \tag{3$'$}$$

or

$$ds^2 = \mathscr{I}^2 d\mathscr{E}^2 + d\mathscr{H}^2 + d\mathscr{I}^2. \tag{3$''$}$$

Proof: By the conditions of Lemma 2., we may take the quantities $\mathscr{E}$, $\mathscr{H}$ and $\mathscr{I}$ as coordinates in a triply orthogonal coordinate system. If we denote the corresponding scale factors by h_1, h_2 and h_3 then, since $|\nabla\mathscr{E}| = h_1^{-1}$, $|\nabla\mathscr{H}| = h_2^{-1}$, and $|\nabla\mathscr{I}| = h_3^{-1}$.
we have

$$h_2 = gh_1, \qquad h_3 = 1. \tag{5}$$

Now the scale factors must satisfy the conditions of Lamé [3]. (These are the conditions for the vanishing of the curvature tensor of an orthogonal line element in three dimensions). For brevity, let a subscript 1, 2, or 3 following a comma denote differentiation with respect to $\mathscr{E}$, $\mathscr{H}$, or $\mathscr{I}$, respectively. Then the Lamé conditions for our case are:

$$\begin{aligned} &h_{1,22}h_1 + h_{2,11}h_2 = \frac{h_2}{h_1}h_{1,1}h_{2,1} + \frac{h_1}{h_2}h_{1,2}h_{2,2} - h_1h_2h_{1,3}h_{2,3}, \\ &h_2h_{1,23} = h_{1,2}h_{2,3}, \qquad h_{1,33} = 0, \\ &h_1h_{2,13} = h_{2,1}h_{1,3}, \qquad h_{2,33} = 0. \end{aligned} \tag{6}$$

We see immediately that h_1 and h_2 have the forms

$$\begin{aligned} h_1 &= \alpha\mathscr{I} + \beta \\ h_2 &= \gamma\mathscr{I} + \delta \end{aligned} \tag{7}$$

where α, β, γ, and δ are functions of $\mathscr{E}$ and $\mathscr{H}$ only. Furthermore we must have from the last two Lamé conditions on the left

$$\alpha\delta_{,1} = \beta\gamma_{,1} \qquad \delta\alpha_{,1} = \gamma\beta_{,2} \tag{8}$$

since they must be true for all $\mathscr{I}$. Now we have

$$(\gamma\mathscr{I} + \delta) = g(\alpha\mathscr{I} + \beta), \tag{9}$$

and differentiation of this with respect to $\mathscr{E}$ or $\mathscr{H}$ yields equations from which g can be eliminated using (9) (since $g \neq 0$). Then identification of coefficients of powers of $\mathscr{I}$ yields linear homogeneous equations in the derivatives of α, β, γ, and δ. If we consider these in conjunction with (8) we find that either (a) $\alpha\delta = \beta\gamma$ or (b) α, β, γ and δ must all be constants.

In the latter case, simple substitution into the first Lamé condition

shows that either $\alpha = 0$, in which case g is a linear function of $\mathscr{I}$, or $\gamma = 0$, so that g^{-1} is linear in $\mathscr{I}$, or both α and γ vanish. If $\alpha = \gamma = 0$, then $g = \delta/\beta = \text{constant}$, which will be considered under case (a). In no event can δ and β vanish simultaneously. Now, if g or g^{-1} is a linear function of $\mathscr{I}$, it is clear that a redefinition of $\mathscr{I}$ in terms of this linear function, and a renormalization of $\mathscr{E}$ and $\mathscr{H}$ will bring the Euclidean metric into one of the forms (3)′ or (3)″. Note that the surfaces $\mathscr{I} = \text{constant}$ for these metrics, are circular cylinders along whose generators either $\mathscr{E}$ or $\mathscr{H}$ varies.

In case (a), we quickly see that

$$\gamma h_1 = \alpha h_2,$$
$$\delta h_1 = \beta h_2$$

so that by (5), $g = \text{constant}$. A rescaling of the variable $\mathscr{H}$ allows us to take this constant equal to *unity*. Hence also

$$\gamma = \alpha,$$
$$\delta = \beta.$$

Equations (8) now imply $\alpha d\beta = \beta d\alpha$; this is satisfied if and only if

$$\alpha = mp(\mathscr{E}, \mathscr{H}),$$
$$\beta = np(\mathscr{E}, \mathscr{H})$$

where m and n are constants. Simple substitution into the first Lamé condition shows that p must satisfy the equation (4).

From (7) we have

$$h_1 = h_2 = (m\mathscr{I} + n)p(\mathscr{E}, \mathscr{H})$$

so that the line element is indeed of the form (3). The situation for which α and γ both vanish and $\beta = \delta = \text{constant}$ is obviously a special case for which $m = 0$ and $p = \text{constant}$.

LEMMA 4. [4] *The surfaces $\mathscr{I} = \text{constant}$ in a triply orthogonal system of surfaces $\mathscr{E} = \mathscr{E}(x, y, z)$, $\mathscr{H} = \mathscr{H}(x, y, z)$, and $\mathscr{I} = \mathscr{I}(x, y, z)$, chosen so that the Euclidean line element can be written in the form*

$$ds^2 = S(\mathscr{I})\{g_1(\mathscr{E}, \mathscr{H})\, d\mathscr{E}^2 + g_2(\mathscr{E}, \mathscr{H})\, d\mathscr{H}^2\} + d\mathscr{I}^2$$

are either spheres or planes.

§4. It is convenient to introduce the notation

$$\begin{aligned} k_1 + il_1 &= \mu(\omega + i\alpha_0)e^{-i\psi}, \\ k_2 + il_2 &= [\varepsilon(\omega + i\alpha_0) - i\sigma]e^{i\psi}, \\ E_0 = |\mathbf{E}_0|, &\qquad H_0 = |\mathbf{H}_0|. \end{aligned} \tag{10}$$

Note that $l_1 = dk_1/d\psi$, $l_2 = -dk_2/d\psi$. Then we may state and prove:

THEOREM 1. *Necessary and sufficient conditions that electromagnetic vectors of the form* (2) *be solutions of the system* (1) *are that*

(i) ψ *is a function of* φ,

$$\text{(ii)}\ \nabla\varphi = \Phi\frac{\mathbf{E}_0 \times \mathbf{H}_0}{E_0 H_0}, \quad \textit{where } \Phi^2 = \left|\frac{k_1 k_2}{1-\psi'}\right|, \tag{11}$$

(iii) $\mathbf{E}_0$ *and* $\mathbf{H}_0$ *are orthogonal vectors for which*

$$E_0^2 = \left|\frac{k_1}{k_2}(1-\psi')\right| H_0^2, \tag{12}$$

(iv) $\mathbf{E}_0$ *and* $\mathbf{H}_0$ *have the forms*

$$\mathbf{E}_0 = \exp\left\{-\int_{\varphi_0}^{\varphi}\frac{l_1}{k_1}d\varphi\right\}\nabla\mathscr{E}, \quad \mathbf{H}_0 = \exp\left\{-\int_{\varphi_0}^{\varphi}\frac{l_2}{k_2}(1-\psi')d\varphi\right\}\nabla\mathscr{H} \tag{13}$$

where φ_0 *is an arbitrary lower limit and* $\mathscr{E}$ *and* $\mathscr{H}$ *are scalar function of position.*

Proof: If we substitute the expressions (2) into (1) and separate the resulting equations into real and imaginary parts, the system becomes

$$\begin{aligned}
\nabla \times \mathbf{E}_0 - l_1\mathbf{H}_0 &= 0, & \nabla\cdot\mathbf{E}_0 &= 0,\\
\nabla \times \mathbf{H}_0 + l_2\mathbf{E}_0 &= 0, & \nabla\cdot\mathbf{H}_0 &= 0,\\
\nabla\varphi \times \mathbf{E}_0 + k_1\mathbf{H}_0 &= 0, & \nabla\varphi\cdot\mathbf{E}_0 &= 0,\\
(\nabla\varphi - \nabla\psi) \times \mathbf{H}_0 - k_2\mathbf{E}_0 &= 0, & (\nabla\varphi-\nabla\psi)\cdot\mathbf{H}_0 &= 0.
\end{aligned} \tag{14}$$

The equations in the left hand column will be termed "vector" equations, and those on the right will be called "scalar" equations.

From these equations we infer the additional orthogonality relations

$$\mathbf{E}_0\cdot\mathbf{H}_0 = \nabla\varphi\cdot\mathbf{H}_0 = \nabla\psi\cdot\mathbf{H}_0 = \nabla\psi\cdot\mathbf{E}_0 = 0.$$

Hence the vectors $\nabla\varphi$, $\nabla\psi$, $\mathbf{E}_0 \times \mathbf{H}_0$ are parallel. Thus ψ is a function of φ and we may write

$$\nabla\psi = \frac{d\psi}{d\varphi}\nabla\varphi = \psi'\nabla\varphi.$$

Also $\nabla\varphi$ has the form (11). For, $|\nabla\varphi|\, E_0 = |k_1|\, H_0$ and $|\nabla\varphi|\, H_0 = |k_2/(1-\psi')|\, E_0$ from the last two vector equations in (14). Hence $|\nabla\varphi|^2 = |k_1k_2/(1-\psi')|$, from which (11) follows immediately. Moreover we see that $E_0/H_0 = |(k_1/k_2)(1-\psi')|^{1/2}$ which is just (12).

We note that $\mathbf{E}_0\cdot\nabla\times\mathbf{E}_0 = \mathbf{H}_0\cdot\nabla\times\mathbf{H}_0 = 0$. Application of Lemma 1 permits us to write

$$\begin{aligned}\mathbf{E}_0 &= e^a\nabla\mathscr{E},\\ \mathbf{H}_0 &= e^b\nabla\mathscr{H}.\end{aligned} \tag{15}$$

Substitution of these expressions into the vector equations in (14) yields

$$\begin{aligned}
\nabla\varphi \times \nabla\mathscr{E} &= -\,k_1\nabla\mathscr{H}e^{b-a},\\
\nabla\varphi \times \nabla\mathscr{H} &= \frac{k_2}{1-\psi'}\,\nabla\mathscr{E}e^{a-b},\\
\nabla a \times \nabla\mathscr{E} &= l_1\nabla\mathscr{H}e^{b-a},\\
\nabla b \times \nabla\mathscr{H} &= -\,l_2\nabla\mathscr{E}e^{a-b}.
\end{aligned}$$

Clearly a is a function of φ and $\mathscr{E}$ only, while b is a function of φ and $\mathscr{H}$ only. Moreover these equations show that

$$\begin{aligned}
\frac{\partial a}{\partial\varphi} &= -\frac{l_1}{k_1},\\
\frac{\partial b}{\partial\varphi} &= -\frac{l_2}{k_2}(1-\psi')
\end{aligned}$$

which are functions of φ only. Hence

$$\begin{aligned}
a &= -\int_{\varphi_0}^{\varphi}\frac{l_1}{k_1}\,d\varphi + a_0(\mathscr{E}),\\
b &= -\int_{\varphi_0}^{\varphi}\frac{l_2}{k_2}(1-\psi')\,d\varphi + b_0(\mathscr{H})
\end{aligned}$$

where φ_0 is any arbitrarily chosen lower limit. By examining the resulting expressions for $\mathbf{E}_0$, $\mathbf{H}_0$ we find that the factors e^{a_0}, e^{b_0} may be absorbed into $\nabla\mathscr{E}$, $\nabla\mathscr{H}$ (by replacing $\mathscr{E}$, $\mathscr{H}$ by $\int e^{a_0}d\mathscr{E}$, $\int e^{b_0}d\mathscr{H}$). Thus we finally have

$$\begin{aligned}
a &= a(\varphi) = -\int_{\varphi_0}^{\varphi}\frac{l_1}{k_1}\,d\varphi,\\
b &= b(\varphi) = -\int_{\varphi_0}^{\varphi}\frac{l_2}{k_2}(1-\psi')\,d\varphi.
\end{aligned} \tag{16}$$

We now note that (13) is a consequence of (15) and (16), which completes the proof of the necessity of Theorem 1.

As an immediate consequence of the above we see that if $\mathscr{I}$ is defined by

$$\mathscr{I} = \int_{\varphi_1}^{\varphi}\left|\frac{1-\psi'}{k_1k_2}\right|^{1/2}d\varphi \tag{17}$$

where φ_1 is arbitrary, the functions $\mathscr{I}$, $\mathscr{E}$, and $\mathscr{H}$ all satisfy the conditions of Lemma 2. For, $g = |\nabla\mathscr{E}|/|\nabla\mathscr{H}|$ is defined by

$$g = \exp\left\{\int_{\varphi_0}^{\varphi}\left[\frac{l_1}{k_1}-\frac{l_2}{k_2}(1-\psi')\right]d\varphi\right\}\left|\frac{k_1}{k_2}(1-\psi')\right|^{1/2} \tag{18}$$

and this is a function of $\mathscr{I}$ through (17). In proving the sufficiency of Theorem 1 by direct differentiation and substitution, use must be made of Lemma 2. We formalize this step in the next result.

THEOREM 2. *The "potential" functions $\mathscr{E}$ and $\mathscr{H}$ of Theorem* 1 *satisfy the equations*

$$\nabla^2\mathscr{E} = \nabla^2\mathscr{H} = 0. \tag{19}$$

The surfaces of constant phase, $\varphi =$ constant, are also surfaces on which $\mathscr{I} =$ constant. Hence, in view of Lemmas 3 and 4, we have the additional

THEOREM 3. *The surfaces of constant phase φ, for electromagnetic waves of the type* (2), *are either concentric circular cylinders, concentric spheres, or parallel planes.*

In proving the above theorem, it will be seen that the function g in (18) is not unrestricted but is either a linear function of $\mathscr{I}$, the reciprocal of one, or a constant. By suitably choosing φ_0 and φ_1 we may infer in any case that

$$\left[\int_{\varphi_1}^{\varphi}\left|\frac{1-\psi'}{k_1k_2}\right|^{1/2}d\varphi\right]^{\pm 1,0} = \left|\frac{k_1}{k_2}(1-\psi')\right|^{1/2}\exp\left\{\int_{\varphi_0}^{\varphi}\left[\frac{l_1}{k_1}-\frac{l_2}{k_2}(1-\psi')\right]d\varphi\right\} \tag{20}$$

are the corresponding three rather complicated-appearing integro-differential equations for ψ. The exponent on the left is $+1$ if g is a linear function of $\mathscr{I}$, so that $\mathscr{E}$ varies along the cylinder generators; it is -1 if g^{-1} is a linear function of $\mathscr{I}$, so that $\mathscr{H}$ varies along the cylinder generators; it is 0 in the case of spherical or plane waves. Simpler forms result if we recall that

$$\frac{d\mathscr{I}}{d\varphi} = \left|\frac{1-\psi'}{k_1k_2}\right|^{1/2}. \tag{20a}$$

Then the foregoing equations (20) can be rewritten, after a logarithmic differentiation in which it must be remembered that $k_1' = l_1\psi'$ and $k_2' = -l_2\psi'$, in the form

$$\frac{d}{d\varphi}\ln\left(\mathscr{I}^{\mp 1,0}\frac{d\mathscr{I}}{d\varphi}\right) = \frac{l_2}{k_2}(1-\psi') - \frac{l_1}{k_1}(1+\psi'). \tag{20b}$$

These results lead to

THEOREM 4. *The phase difference function $\psi(\varphi)$ in Theorem* 1 *must satisfy the integro-differential equations* (20), *or the differential equations* (20a, b).

The application of Lemma 3 in the proof of Theorem 3 may also be formalized by

THEOREM 5. *A necessary and sufficient condition that the functions $\mathscr{E}$, $\mathscr{H}$, φ of Theorem* 1 *exist is that the line element in physical* (*Euclidean*) *space*

is expressible either in one of the two forms (3)'

(i) $ds^2 = d\mathcal{E}^2 + \mathcal{I}^2 d\mathcal{H}^2 + d\mathcal{I}^2$

(*for cylindrical waves with the electric vector parallel to the generators*) *and* (3)''

(ii) $ds^2 = \mathcal{I}^2 d\mathcal{E}^2 + d\mathcal{H}^2 + d\mathcal{I}^2$

(*for cylindrical waves with the magnetic vector parallel to the generators*) *or in the form* (3)

(iii) $ds^2 = p^2(\mathcal{E}, \mathcal{H})(m\mathcal{I} + n)^2(d\mathcal{E}^2 + d\mathcal{H}^2) + d\mathcal{I}^2$

(*for spherical and plane waves*) *where* $\mathcal{I}$ *is given by* (17), p *satisfies* (4), *and* $m = 0$ *for plane waves, or* $m \neq 0$ *for spherical ones.*

As we shall see, this last theorem suggests a convenient method for writing down or generating various classes of solutions of Maxwell's equations for spherical and plane waves.

§5. The foregoing theorems bring out several points of physical and mathematical interest. If, for simplicity one fixes attention on the usual plane wave solutions, in which the phase difference ψ is a constant, then it can be seen that such solutions correspond to a singular solution of the non-linear integro-differential equation in (20) with the exponent 0, namely

$$\psi = \frac{c + d}{2} \tag{21}$$

where

$$\begin{aligned} c &= -\arg\,[\varepsilon(\omega + i\alpha_0) - i\sigma], \\ d &= \arg\,[\mu(\omega + i\alpha_0)]. \end{aligned} \tag{22}$$

The interesting question then arises: if the phase difference of a solution differs slightly from (21) for some particular φ-value, will the perturbation decay or grow as one proceeds in the direction of propagation toward *decreasing* φ-values? Put in other terms, is the phase difference of the usual plane wave solutions in stable or unstable equilibrium?

An elementary investigation shows that the latter alternative is the case in ordinary conducting media. For simplicity let us take σ, ε, μ as real and $\alpha_0 = 0$. Then $d = 0$ and $c = \tan^{-1} \sigma/\omega\varepsilon$. Now if π is a small deviation of ψ from the value $c/2$, π must satisfy the differential equation

$$\pi'' + 4\pi' \tan\frac{c}{2} + 4\pi\left(1 + \tan^2\frac{c}{2}\right) = 0. \tag{23}$$

Since φ increases *negatively* in the direction of propagation, π is an oscillating increasing function in this direction, and the singular solution is unstable. This peculiar behaviour of the phase difference between the **E** and **H** vectors is to be investigated in considerably more detail in another paper, in which the physical consequences are discussed. We merely point out here: 1) the

restriction to plane waves is not essential to the above argument, 2) the form of the vectors $\mathbf{E}_0$ and $\mathbf{H}_0$ indicates that the amplitudes of these waves also depend in a curious way on the phase, and 3) the phase difference behaves differently for cylindrical waves as compared to the other types. In particular, for cylindrical waves of our type no solution for which $\psi =$ constant seems to exist.

The next point of interest is that in the case of spherical or plane waves, the curves $\mathscr{E} =$ const and $\mathscr{H} =$ const form an isometric orthogonal net on surfaces of constant phase φ, but are otherwise rather general. Moreover if one pair of functions $(\mathscr{E}, \mathscr{H})$, and a solution $p\ (\mathscr{E}, \mathscr{H})$ of 4 are known all other such pairs and solutions may be determined by the use of harmonic functions [5]. For, let w be any harmonic function of $\zeta = \mathscr{E} + i\mathscr{H}$. Then a new pair $(\mathscr{E}', \mathscr{H}')$ is determined by

$$\mathscr{E}' + i\mathscr{H}' = w(\mathscr{E} + i\mathscr{H}) \tag{24}$$

and the function

$$p'(\mathscr{E}', \mathscr{H}') = p(\mathscr{E}, \mathscr{H})\ |\ dw/d\zeta\ |^{-1} \tag{25}$$

is also a solution of an equation of the form (4), written in terms of $\mathscr{E}'$ and $\mathscr{H}'$. This is easily checked by direct substitution. For any such solution $p(\mathscr{E}, \mathscr{H})$ of (4), the magnitudes of the gradients of $\mathscr{E}$ and $\mathscr{H}$ are of course given by

$$|\ \nabla\mathscr{E}\ | = |\ \nabla\mathscr{H}\ | = p^{-1}(\mathscr{E}, \mathscr{H})(m\mathscr{J} + n)^{-1} \tag{26}$$

where $\mathscr{J}(x, y, z) =$ constant is a suitable sphere or, if $m = 0$, a plane. In addition, if $\mathbf{i}_{\mathscr{E}}$ and $\mathbf{i}_{\mathscr{H}}$ are unit vectors in the directions of the old electric and magnetic vectors respectively, and we write $dw/d\zeta = |dw/d\zeta| \exp\{i\eta\}$, then for the new solution, the corresponding new directions are given by

$$\begin{aligned} \mathbf{i}_{\mathscr{E}'} &= \mathbf{i}_{\mathscr{E}} \cos\eta - \mathbf{i}_{\mathscr{H}} \sin\eta, \\ \mathbf{i}_{\mathscr{H}'} &= \mathbf{i}_{\mathscr{E}} \sin\eta + \mathbf{i}_{\mathscr{H}} \cos\eta. \end{aligned} \tag{27}$$

Thus we see that the factors $\nabla\mathscr{E}$ and $\nabla\mathscr{H}$ appearing in the solution (13) of Maxwell's equations are determinable for our spherical and plane wave cases by the methods of conformal mapping. For the cylindrical wave case we obtain only solutions for which essentially

$$\begin{array}{lll} \text{either} & |\nabla\mathscr{E}| = \mathscr{J}^{-1}, & |\nabla\mathscr{H}| = 1 \\ \text{or} & |\nabla\mathscr{H}| = \mathscr{J}^{-1}, & |\nabla\mathscr{E}| = 1. \end{array} \tag{28}$$

In order to calculate explicitly the exponential factors in (13), a solution $\psi(\varphi)$ of equations (20) must be found, and then φ must be determined in terms of $\mathscr{J}$ by inversion of (17).

§6. We conclude with several elementary illustrative solutions of Maxwell's equations in the form (13). In these examples we shall not concern outselves with a general determination of $\varphi(x, y, z)$, leaving this for a later paper, but will consider only solutions for which $\psi =$ constant. This automatically excludes the case of cylindrical waves. In such cases we may take, with $\varphi_0 = \varphi_1$,

$$\varphi = \sqrt{|k_1 k_2|}\, \mathscr{I} + \varphi_0 \tag{29}$$

so that

$$\mathbf{E}_0 = \nabla\mathscr{E} \exp\{-l_1|k_2/k_1|^{1/2}\mathscr{I}\}, \qquad \mathbf{H}_0 = \nabla\mathscr{H} \exp\{-l_2|k_1/k_2|^{1/2}\mathscr{I}\} \tag{30}$$

where l_1, l_2, k_1, k_2 are all constant. Hence we need to determine only the explicit forms of $\mathscr{I}$, $\nabla\mathscr{E}$, and $\nabla\mathscr{H}$.

As a first example consider a cylindrical coordinate system (ϱ, θ, z), in which

$$ds^2 = d\varrho^2 + \varrho^2\, d\theta^2 + dz^2. \tag{31}$$

Rewrite this as

$$ds^2 = \varrho^2[(d \ln \varrho)^2 + d\theta^2] + dz^2. \tag{32}$$

Apparently we may take $m = 0$, $n = 1$, $p = \varrho$ and

$$\begin{aligned} \mathscr{E} &= \ln \varrho, \\ \mathscr{H} &= \theta, \\ \mathscr{I} &= z. \end{aligned}$$

Then the electric and magnetic vectors, by formulas (26) and (30), are proportional respectively to

$$\begin{aligned} \nabla\mathscr{E} &= \mathbf{i}_\varrho\, \varrho^{-1}, \\ \nabla\mathscr{H} &= \mathbf{i}_\theta\, \varrho^{-1} \end{aligned}$$

and the waves travel along the z-axis. This corresponds to the well-known solution for plane waves travelling along a wire conductor coincident with the z-axis.

As the second example, consider a spherical coordinate system (r, θ, λ) in which the line element may be written as

$$ds^2 = dr^2 + r^2 \sin^2 \lambda \left[(d\theta)^2 + \left(\frac{d\lambda}{\sin \lambda}\right)^2 \right].$$

Then we may take $m = 1$, $n = 0$, $p = \sin \lambda$ and

$$\begin{aligned} \mathscr{E} &= \ln \tan \frac{\lambda}{2}, \\ \mathscr{H} &= \theta, \\ \mathscr{I} &= r, \end{aligned}$$

so that the waves are travelling radially and the electric and magnetic vectors are both proportional to $1/r \sin \lambda$, but point in the polar and azimuthal directions respectively. Imagine a perfectly conducting surface, in the shape of an infinite right circular cone, to be placed with apex at the origin, and axis along the z-axis. Also suppose a perfectly conducting wire to coincide with the cone axis. Then our solution corresponds to electromagnetic waves traveling along the wire in the space between it and the cone surface.

In this example, as in the previous one, the direction of either of the two symmetric isometric coordinates might have been used to correspond to the direction of the electric vector, the other to the magnetic vector, with a corresponding change in interpretation of boundary conditions.

§7. The authors wish to express their appreciation to Professors W. Magnus and F. Reiche for carefully reading and checking the results contained herein and for giving them many useful suggestions.

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Received November, 1954.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. IX, 45–68 (1956)

On the Non-Existence of Continuous Transonic Flows Past Profiles I*

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In this paper we shall show that the perturbation problem belonging to a two-dimensional steady transonic flow past an obstacle is not correctly posed. This theorem has been proposed in various forms[1] [1, 2, 3, 4] as an explanation for the breakdown of continuous transonic flow. No rigorous proof has been given before. The statement of the theorem may be found as Conjectured Theorem C in [1]. The basic plausibility arguments were developed by Busemann [2], Frankl [3] and Guderley [4].

The perturbation problem may be described roughly as follows. Suppose for some Mach number at infinity, $M_\infty < 1$, there is a steady continuous symmetric transonic flow, past a profile given by $y = \pm\, Y(x)$, with continuously differentiable potential φ and stream function ψ. Analytic expressions describing such flows are known, see for example the work of Lighthill, Cragg and Goldstein, or Tomatika and Tamada. φ and ψ form a solution of a boundary value problem for a pair of nonlinear elliptic-hyperbolic equations of first order and ψ vanishes on $y = \pm\, Y(x)$.

Consider any neighboring profile $P + \delta P$ given by $y = \pm\,(Y(x) + \delta Y(x))$. If the boundary value problem described above is properly set there should exist, for general variations of the profile $\delta Y(x)$, a flow with stream function $\psi + \delta\psi$ and potential $\varphi + \delta\varphi$, having the same Mach number and flow direction at infinity, satisfying $\psi + \delta\psi = 0$ on $y = \pm\,(Y(x) + \delta Y(x))$ and for which $\delta\psi$, $\delta\varphi$ and their derivatives are small of the order of δY. Such is the case if the original flow is everywhere subsonic provided δY is sufficiently smooth. For mixed flows we know that we cannot vary the profile by putting a concave piece into the supersonic region, see for example [5, p. 294]. Nor can we have a flat piece [6] or a piece with zero curvature [7] in the supersonic profile. These conditions limit the admissible functions $\delta Y(x)$ but we shall show that much more is true.

We shall prove within the framework of the perturbation theory that

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*The results of this paper were obtained under the sponsorship ot the U.S. Army, Office of Ordnance Research, Contract DA-30-069-ORD-835 and were announced in Abstract 433, Bulletin of the American Mathematical Society, Volume 60, No. 4, July 1954.

[1]See [1] for a more complete bibliography.

for convex symmetric bodies[3] *the variation of the profile can be prescribed at most outside some finite arc containing the point of maximum velocity.* Except for an arbitrary constant $\delta Y(x)$ on the arc is determined by the variation $\delta Y(x)$ outside the arc.

The proof is based on a uniqueness theorem. $\delta\varphi$ and $\delta\psi$ satisfy a pair of mixed elliptic-hyperbolic equations of first order. If terms of order greater than one in $\delta\varphi_x$, $\delta\varphi_y$, $\delta\psi_x$, $\delta\psi_y$ are neglected, these equations are linear with coefficients depending only on φ_x, φ_y. Furthermore $\delta\varphi_x$, $\delta\varphi_y$ vanish at infinity and the value of $\delta\psi$ is determined on the profile by $\delta Y(x)$. We prove that, except for two constants, $\delta\varphi$ and $\delta\psi$ are uniquely determined everywhere by the value of $\delta Y(x)$ on *part* of the profile, so that $\delta Y(x)$ is determined on the rest of the profile.

We would have a rigorous proof that continuous transonic flows past profiles do not exist in general if the same uniqueness theorem could be proved without neglecting the higher order terms in $\delta\varphi$ and $\delta\psi$, see Conjectured Theorem A [1].

In what follows we formulate the problem not in terms of the variations $\delta\varphi$ and $\delta\psi$ but, completely equivalently, by considering a set of flows, with the same Mach number and flow direction at infinity, past profiles depending differentiably on a parameter τ and also on an arbitrary function. The equivalent non-existence theorem is that the velocities can not depend differentiably on τ except for a particular choice of the arbitrary function. This demonstrates that there do not exist continuous transonic flows with the smoothness properties one might have expected from a study of the behavior of subsonic flows.

I am very much indebted to L. Bers for his valuable criticism and many suggestions. In particular, the treatment of the point at infinity is due to him.

1. Formulation of the Problem

Consider a given smooth, two-dimensional, steady, irrotational, compressible flow $\mathscr{F}_0$, with potential $\varphi(x, y)$, stream function $\psi(x, y)$, and complex velocity $u - iv$, past a smooth convex profile P_0. The free stream Mach number M_0 is less than 1 but there is a finite supersonic region next to the profile. The flow $\mathscr{F}_0$ will be referred to as the unperturbed flow. The profile and flow are symmetric and we shall consider only the upper half of the x, y-plane. The flow $\mathscr{F}_0$ is subsonic except in some finite neighborhood of the

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[3] Manwell [8] has shown for mixed flow around a cylinder with zero velocity at infinity that there are infinitely many variations of the cylinder which admit continuous solutions of the perturbation problem. But they are such that this boundary value problem is also badly set.

profile where it is supersonic. The curve dividing the subsonic and the supersonic region is called the sonic curve. It intersects the profile at X_1 and X_2. Let S be the segment of the profile cut out by two Mach lines, C_1 and C_2, issuing from a point X on the sonic curve (see Figure 1). S contains the point of maximum velocity on the profile. The end points of S are denoted by X_3 and X_4.

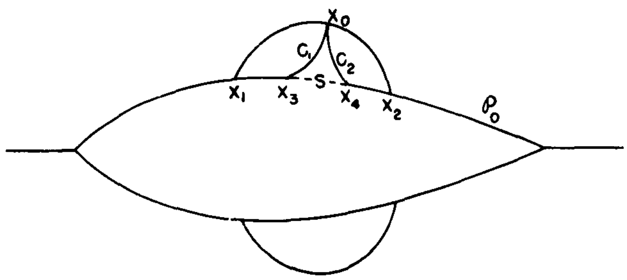

Figure 1.

Let $\Phi(x, y, \tau)$ and $\Psi(x, y, \tau)$ be the velocity potential and stream function of a family of irrotational steady symmetric continuous flows $\mathscr{F}_\tau$, depending *differentiably* on τ, with the same free stream Mach number as $\mathscr{F}_0$, around a set of profiles $P(\tau)$ that coincide with P_0 except on S. For $\tau = 0$ the flow is $\mathscr{F}_0$. The functions $\omega(x, y) = \frac{\partial}{\partial\tau}\Phi(x, y, \tau)|_{\tau=0}$ and $\omega^*(x, y) = \frac{\partial}{\partial\tau}\Psi(x, y, \tau)|_{\tau=0}$ are the perturbation potential and stream function, respectively, and are defined in the domain $\mathscr{E}(P)$ exterior to P_0. It is assumed that $\Phi_x, \Phi_y, \Psi_x, \Psi_y$ are differentiable with respect to τ and that τ-differentiation commutes with differentiation in the x and y directions.

The potential Φ and stream function Ψ are related by the equations

$$(1)\qquad \begin{aligned} -\Psi_y &= \varrho\Phi_x, \\ \Psi_x &= \varrho\Phi_y. \end{aligned}$$

The density ϱ is a given function of velocity, $\varrho = \varrho(|w|^2)$, $|w|^2 = \Phi_x^2 + \Phi_y^2$. The functions ω and ω^* satisfy (1) differentiated with respect to τ at $\tau = 0$; that is, in $\mathscr{E}(P)$,

$$(2)\qquad \begin{aligned} -\omega_y^* &= \varrho(\omega_x + 2m\varphi_x(\varphi_x\omega_x + \varphi_y\omega_y)), \\ \omega_x^* &= \varrho(\omega_y + 2m\varphi_y(\varphi_x\omega_x + \varphi_y\omega_y)), \end{aligned}$$

where $m = \frac{1}{\varrho}(d\varrho/d\,|w|^2)\,|_{\tau=0}$ and $\varrho = \varrho(|w|^2)\,|_{\tau=0}$.

The function m is given by Bernoulli's equation, $qdq + \varrho^{-1}c^2\,d\varrho = 0$,

where q is the velocity of the unperturbed flow, that is,

$$m = -\frac{1}{2c^2} \tag{3}$$

and c is the speed of sound. We assume that

$$\frac{dm}{d\,|w|^2} = \frac{1}{2c^4}\, dc^2/d\,|w|^2 < 0 \tag{4}$$

which is true for all gases whose pressure-density relation satisfies $d^2p/d\varrho^2 > 0$.

The boundary condition is $\Psi = 0$ on P and for $y = 0$, and hence

$$\omega^* = 0, \qquad \psi = 0, \tag{5}$$

on $P_0 - S$ and on the x-axis.

On the remainder of the profile we set $y = Y(x, \tau) = Y_0(x) + \tau Y_1(x) + \cdots$ where $Y_1(x)$ is a function that vanishes at the end points of S, X_3 and X_4. The boundary condition on S is $\Psi(x, y, \tau) = 0$. Differentiating this condition with respect to τ at $\tau = 0$ yields

$$\omega^* + \psi_y Y_1 = 0 \qquad \text{on } S. \tag{6}$$

The potential φ of the unperturbed flow $\mathscr{F}_0$ is assumed to be smooth: First the third derivatives of φ all exist in $\mathscr{E}(P)$ and the second derivatives are Hölder continuous on the boundary except at the nose and trailing edge. Secondly, the upper half of the domain outside P is mapped, continuously differentiably except at the nose, the tail and at ∞, in a one-to-one way into the u, v-plane where $u = \varphi_x(x, y)$, $v = \varphi_y(x, y)$ and the Jacobian

$$x_u y_v - x_u y_v \neq 0 \qquad \text{except at } \infty. \tag{7}$$

We also assume that on the profile in the supersonic region

$$\frac{dq}{d\theta} \text{ is monotonic}, \quad q^2 \geqq c^2. \tag{8}$$

The functions φ corresponding to transonic flows computed by the hodograph method have these properties.

At infinity we have

$$\Phi_x - i\Phi_y \to q_\infty > 0 \tag{9}$$

where q_∞ is the speed at infinity.

Since q_∞ is assumed to be independent of τ, we have

$$\omega_x - i\omega_y \to 0 \qquad \text{at } \infty. \tag{10}$$

Under these conditions, we can prove the following lemma.

LEMMA 1. *If ω and ω^* satisfy* (2), (5), (10) *in $\mathscr{E}(P)$ and have continuous derivatives for $x^2 + y^2 > R^2$ then*

$$(1 - M_\infty^2)^{-\frac{1}{2}}\omega_u - i\omega_v = b(u - q_\infty - iv)^{-\frac{1}{2}}(1 + o(1)) \tag{11}$$

where b is a constant.

If we assume that the flow is incompressible for speeds close to q_∞, Lemma 1 is easily proved. First $\Phi_x - i\Phi_y - q_\infty$ is analytic in $x + iy$ for large $x^2 + y^2$. Therefore, by the condition of symmetry and the single-valuedness of the stream function we find

$$\Phi_x - i\Phi_y - q_\infty = \frac{a_0(\tau)}{(x+iy)^2}\left(1 + \frac{a_1(\tau)}{x + iy} + \cdots\right) \tag{12}$$

where $a_0, a_1, \cdots$ are differentiable with respect to τ. Differentiating (12) at $\tau = 0$, which is easily justified, we have

$$\omega_x - i\omega_y = \frac{a_{0\tau}(0)}{(x + iy)^2}(1 + O((x + iy)^{-1}). \tag{13}$$

Inverting (12) for $\tau = 0$ and differentiating with respect to $u = iv - q_\infty$, we find

$$\frac{d(x + iy)}{d(u - iv - q_\infty)} = \frac{a_0^{1/2}(0)}{2(u - iv - q)^{3/2}}(1 + o(1)).$$

Therefore

$$\omega_u - i\omega_v = \frac{a_{0\tau}(0)a_0^{-1/2}(0)}{2(u - iv - q_\infty)^{1/2}}(1 + o(1)).$$

This yields (11) since $M_\infty = 0$.

For general compressible flow Lemma 1 is established in Appendix 1, by using the corresponding knowledge for compressible flows given by Bers in [11].

The flows $\mathscr{F}_\tau$ are assumed to have continuous velocities and piecewise continuous accelerations, except at X_0, X_1 and X_2, which are differentiable with respect to τ at $\tau = 0$; that is,

(14) ω has continuous derivatives and piecewise continuous second derivatives in $\mathscr{E}(P) - X_0 - X_1 - X_2$.

It follows from (14) and (2) that the same condition holds for ω^*. It is natural to require that the perturbation velocities ω_x, ω_y remain bounded at X_0, X_1 and X_2, but actually our argument holds if we merely assume

(15) ω_x and ω_y are $o(d^{-\varkappa})$ where d is distance from X_0, X_1 or X_2. $\varkappa$ is 1 for X_0 and 1/8 for X_1 and X_2.

It has been conjectured that the perturbation problem might be correctly posed if a sufficiently strong singularity were permitted at X_1 or X_2. Condition (10) indicates what least order of singularity is necessary.

Let us call a pair of functions (ω, ω^*) a solution of Problem I if they satisfy (2), (5), (6), (10), (14), (15). From Lemma 1, ω also satisfies (11). The statement that the perturbation problem is not correctly posed means that there does not exist a solution of Problem I except for particular choices of the functions $Y_1(x)$. We shall prove in fact the following theorem.

THEOREM I. *There is, except for a constant factor, at most one function $Y_1(x)$ such that a solution of Problem I exists.*

The function ω will be called a solution of Problem II if it satisfies (2), (5), (14), (15) and

$$(13)^* \qquad \omega_u - i\omega_v = O\big((u - q_\infty - iv)^{-1/2}\big) \text{ as } x^2 + y^2 \to \infty.$$

Note that no boundary condition is imposed on S.

We shall prove:

THEOREM[3] II. *The only solution of Problem II is $\omega \equiv$ constant, $\omega^* \equiv 0$.*

Theorem II implies Theorem I.

In fact, suppose Problem I has two solutions corresponding to $Y_1 = Y_{11}(x)$, $Y_1 = Y_{12}(x)$. Let the two solutions be (ω_1, ω_1^*) and (ω_2, ω_2^*) respectively and let the constants b in (11) be b_1 and b_2 respectively. We may assume that $\omega_1 = \omega_2 = 0$ at ∞. If b_1 or b_2 is zero, then (ω_1, ω_1^*) or (ω_2, ω_2^*) is a solution of Problem II. If b_1 and b_2 are different from zero, consider the pair of functions $(b_2\omega_1 - b_1\omega_2, b_2\omega_1^* - b_1\omega_2^*)$. They satisfy (2), (5), (14) and (15) and also (13)* at ∞. Hence they form a solution of Problem II. By Theorem II, $b_2\omega_1^* = b_1\omega_1^*$ and by (6), $b_2Y_{11} = b_1Y_{12}$.

The main theorem, Theorem II, is a uniqueness theorem for a system of equations (2) of mixed type with a homogeneous boundary condition (5).

2. Problem II in a Modified Hodograph Plane

Theorem II is proved in the θ, σ-plane where θ and σ are certain simple functions of the unperturbed velocities.

(a) *Transformation of the differential equations.*

First we consider Problem II in the hodograph plane of $\mathscr{F}_0$. We introduce the unperturbed velocities

$$(16) \qquad \begin{aligned} u &= \varphi_x(x, y), \\ v &= \varphi_y(x, y), \end{aligned}$$

[3]The relation between Theorem II and the non-existence statement was pointed out by L. Bers.

as independent variables. Equations (2) become

$$(17)\qquad \begin{aligned} -\omega_u^*\varphi_{xy} - \omega_v^*\varphi_{yy} &= \varrho\omega_u\big((1+2mu^2)\varphi_{xx} + 2muv\varphi_{xy}\big) \\ &\quad + \varrho\omega_v\big((1+2mu^2)\varphi_{xy} + 2muv\varphi_{yy}\big), \\ \omega_u^*\varphi_{xx} + \omega_v^*\varphi_{xy} &= \varrho\omega_u\big((1+2mv^2)\varphi_{xy} + 2muv\varphi_{xx}\big) \\ &\quad + \varrho\omega_v\big((1+2mv^2)\varphi_{yy} + 2muv\varphi_{xy}\big) \end{aligned}$$

or, by solving for ω_u^* and ω_v^*,

$$(18)\qquad \begin{aligned} (\varphi_{xx}\varphi_{yy}-\varphi_{xy}^2)\omega_u^* &= \varrho\omega_u[\varphi_{xy}\{(1+2mu^2)\varphi_{xx}+2muv\varphi_{xy}+(1+2mv^2)\varphi_{yy}\} \\ &\quad +2muv\varphi_{xx}\varphi_{yy}]+\varrho\omega_v[\varphi_{yy}\{4muv\varphi_{xy}+(1+2mv^2)\varphi_{yy}\} \\ &\quad +(1+2mu^2)\varphi_{xy}^2], \\ (\varphi_{xx}\varphi_{yy}-\varphi_{xy}^2)\omega_v^* &= -\varrho\omega_u[\varphi_{xx}\{(1+2mu^2)\varphi_{xx}+4muv\varphi_{xy}\}+(1+2mv^2)\varphi_{xy}^2] \\ &\quad -\varrho\omega_v[\varphi_{xy}\{(1+2mu^2)\varphi_{xx}+2muv\varphi_{xy}+(1+2mv^2)\varphi_{yy}\} \\ &\quad +2muv\varphi_{yy}\varphi_{xx}]. \end{aligned}$$

But for $\tau = 0$ we have from (1) differentiated with respect to x at $\tau = 0$,

$$(1 + 2mu^2)\varphi_{xx} + 4muv\varphi_{xy} + (1 + 2mv^2)\varphi_{yy} = 0.$$

Equation (18) reduces to

$$\begin{aligned} (\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2)\omega_u^* &= \varrho(\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2)[2muv\omega_u - (1 + 2mu^2)\omega_v], \\ (\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2)\omega_v^* &= \varrho(\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2)[(1 + 2mv^2)\omega_u - 2muv\omega_v]. \end{aligned}$$

Since the Jacobian $\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 = u_xv_y - u_yv_x$ does not vanish except at ∞ by (7) we find using (3)

$$(19)\qquad \begin{aligned} \varrho^{-1}c^2\omega_u^* &= -uv\omega_u - (c^2 - u^2)\omega_v, \\ \varrho^{-1}c^2\omega_v^* &= (c^2 - v^2)\omega_u + uv\omega_v\,. \end{aligned}$$

If we introduce as independent variables the angle

$$(20)\qquad \theta = \tan^{-1}\frac{v}{u}$$

and the speed $q = \sqrt{u^2 + v^2}$, equations (19) can be reduced to

$$(21)\qquad \begin{aligned} c^2q\omega_q^* &= \varrho(c^2 - q^2)\omega_\theta\,, \\ \omega_\theta^* &= -\varrho q\omega_q\,. \end{aligned}$$

These equations have been derived by Manwell in [8].

Note that the equations satisfied by the perturbation stream function

and potential are very similar to the equations for the stream function ψ and potential φ as functions of θ and q:

$$c^2 q\varphi_q = \varrho^{-1}(c^2 - q^2)\psi_\theta\,,$$
$$\varphi_\theta = -\varrho^{-1} q\psi_q\,.$$

The perturbation stream function replaces the potential, the perturbation potential replaces the stream function and ϱ^{-1} replaces ϱ.

Next we set

$$\sigma = -\int_{c^*}^{q} \frac{dq}{\varrho q} = -\int_{c^*}^{q} \frac{dq}{q} \exp\left\{\int_0^q \frac{q}{c^2}\,dq\right\}, \tag{22}$$
$$K(\sigma) = \varrho^2 \frac{(c^2 - q^2)}{c^2} = \frac{c^2 - q^2}{c^2} \exp\left\{-2\int_0^q \frac{q}{c^2}\,dq\right\}$$

where c^* is the value of q for which $q = 0$. We assume without loss of generality that the stagnation density is 1. Note that for small q, $\sigma \sim -\log q$.

Equations (21) then reduce to

$$\omega_\sigma^* = -K\omega_\theta\,, \tag{23}$$
$$\omega_\theta^* = \omega_\sigma\,.$$

Note that $K(\sigma)$ has the properties

$$\sigma K(\sigma) \geqq 0 \tag{24}$$
$$\frac{dK}{d\sigma} > 0 \qquad \text{for } \sigma \gtreqless 0.$$

which we shall need later. For, by (22), σ and $K(\sigma)$ vanish only for $q = c^*$. Also

$$\frac{dK}{d\sigma} = \frac{2\varrho q^2}{c^2}\left(1 - \frac{q^2 - c^2}{c^2} - \frac{q^2}{c^2}\frac{dc^2}{dq^2}\right.\exp\left\{-2\int_0^q \frac{q}{c^2}\,dq\right\}$$

and

$$\frac{dc^2}{dq^2} = \left.\frac{dc^2}{d(|w|)^2}\right|_{\tau=0} < 0 \qquad \text{by (4)}$$

and

$$(c^2 - q^2)/K > 0 \qquad \text{by (22).}$$

From (23) differentiated with respect to θ we obtain an equation for ω,

$$K\omega_{\theta\theta} + \omega_{\sigma\sigma} = 0. \tag{25}$$

The differentiation can be easily justified from our assumptions.

(b) *Transformation of the boundary condition.*

From (5) and (23) we have

$$-\,d\omega^* = K\omega_\theta d\sigma - \omega_\sigma d\theta = 0 \tag{26}$$

on $P_0 - S$ and the x-axis and hence also on $\Pi - \Sigma$ and Λ, the images of $P_0 - S$ and the x-axis in the θ, σ-plane.

Note that the function ω satisfies an equation of mixed type similar to the Tricomi equation, with a homogeneous boundary condition on an open arc.

Alternatively from (23), ω^* is a solution of the equation

$$K\omega^*_{\theta\theta} + \omega^*_{\sigma\sigma} - \frac{1}{K}\frac{dK}{d\sigma}\omega^*_\sigma = 0 \tag{27}$$

and the boundary condition (5) reduces to

$$\omega^* = 0 \qquad \text{on } \Pi - \Sigma \text{ and } \Lambda. \tag{28}$$

The uniqueness of solutions of (23) which are constant on such open arcs as $\Pi - \Sigma$ has been studied in [9, 10] but the uniqueness theorem for (23) and (26) or (27) and (28) has not been investigated.

(c) *The mapping in the θ, σ-plane.*

The image of the exterior of half of the profile P_0, $\mathscr{E}(P_0)$, and the x-axis in the θ, σ-plane is indicated in Figure 2. It consists of (a) Ω which

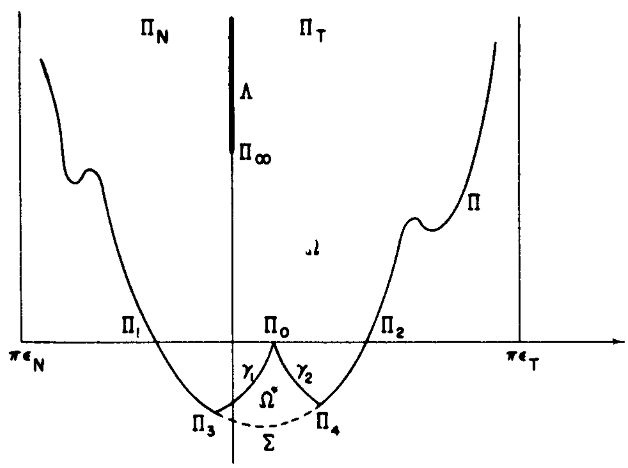

Figure 2.

is bounded by Π, the image of $P_0 - S$ and Λ the image of the x-axis, and the images γ_1 and γ_2 of C_1 and C_2. (b) Ω^* which is bounded by Σ, the image of S and γ_1 and γ_2.

The nose and the trailing edge of the profile P_0 are stagnation points and therefore correspond to $\sigma = +\infty$. Suppose the half angle at the nose is ε_N and at the trailing edge ε_T. Since the profile is convex the domain $\Omega + \Omega^*$ lies between $\theta = \varepsilon_T \Pi$ and $\theta = -\varepsilon_N \Pi$. We denote by Π_T the point $\sigma = +\infty$, $0 \leqq \theta \leqq \varepsilon_T$ and by Π_N the point $\sigma = +\infty$, $-\varepsilon_N \leqq \theta \leqq 0$. The point Π_∞, $\theta = 0$, $\sigma = \sigma_\infty = \sigma(q_\infty)$ is the image of the point at infinity in the physical plane. The σ-axis for $\sigma > \sigma_\infty$ corresponds to the x-axis outside P_0. The axis $\sigma = 0$ is the sonic curve. The points X_0, X_1, X_2, X_3, X_4 are mapped into Π_0, Π_1, Π_2, Π_3, Π_4.

The arc Π satisfies the important condition

$$K d\sigma^2 + d\theta^2 \geqq 0. \tag{29}$$

This expression can be negative only for $\sigma < 0$. But by (5), if $K d\sigma^2 + d\theta^2$ vanishes on Π, then $K\psi_\theta^2 + \psi_\sigma^2$ also vanishes, hence by the Chaplygin equations and equation (22),

$$\begin{aligned} x_u y_v - x_v y_u &= \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial(\varphi, \psi)}{\partial(\theta, \sigma)} \cdot \frac{\partial(\theta, \sigma)}{\partial(u, v)} \Big/ \frac{\partial(\varphi, \psi)}{\partial(x, y)} \\ &= \varrho^{-3}(K\psi_\theta^2 + \psi_\sigma^2)(\varphi_x^2 + \varphi_y^2)^{-1} \frac{\partial(\theta, \sigma)}{\partial(u, v)} = 0 \end{aligned}$$

since $\partial(\theta, \sigma)/\partial(u, v)$ is bounded. But this violates condition (7).

Since γ_1 and γ_2 are characteristics the slopes of γ_1 and γ_2 are given by the equations

$$\begin{aligned} d\theta &= \sqrt{-K}\, d\sigma \qquad \text{on } \gamma_1, \\ d\theta &= -\sqrt{-K}\, d\sigma \qquad \text{on } \gamma_2. \end{aligned} \tag{30}$$

Since the profile is convex we have $d\theta > 0$ on $\Pi - \Sigma$.

(d) *Behavior of derivatives.*

By (9) the derivatives ω_x and ω_y are continuous in $\mathscr{E}(P)$ except at X_0, X_1, X_2 and at infinity and therefore as functions of θ, σ they are continuous except at Π_0, Π_1, Π_2, Π_N, Π_T and Π_∞. The functions x_θ, x_σ, y_θ, y_σ are also continuous except at Π_N, Π_T and Π_∞ since the mappings $(x, y) \to (u, v) \to (\theta, \sigma)$ are continuously differentiable except at these points. We have then

(31) ω_θ and ω_σ are continuous except at Π_0, Π_1, Π_2, Π_N, Π_T and Π_∞.

We shall need a condition on ω_σ at Π_0, Π_3 and Π_4. By (15) ω_x and ω_y are $o(d^{\varkappa_i})$ where d is the distance from X_0, X_1 and X_2 respectively. The functions x_u, x_v, y_u, y_v are boundedat these points. H ence ω_u and ω_v and

therefore, by (20) and (22), ω_θ and ω_σ are $o(d^\varkappa)$. Since the mappings $(x, y) \rightleftarrows (u, v) \rightleftarrows (\theta, \sigma)$ are differentiable at X_0, X_1, and X_2 it follows that

$$\omega_\theta\,,\ \omega_\sigma \text{ are } o\big((\sigma^2 + (\theta - \theta_i)^2)^{-\varkappa_i/2}\big) \text{ where } \varkappa_i = 1, \quad \theta_i = \theta_0 \text{ at } \Pi_0,\ \varkappa_i = 1/8,\ \theta_i = \theta_1,\ \theta_2 \text{ at } \Pi_1\,,\ \Pi_2. \tag{32}$$

We next determine the behavior of the derivatives ω_θ and ω_σ at Π_N and Π_T. These two points correspond to the nose and the trailing edge where the velocity is always zero. (By the convexity condition on P_0 the angle at the trailing edge is finite). It can be shown by the use of the asymptotic expansions at the nose and tail for subsonic flow given in [11] (see Appendix 1 for details) that

$$\omega_\theta e^\sigma,\ \omega_\sigma e^\sigma \text{ are bounded.} \tag{33}$$

We verify (33) here, assuming for simplicity that the flow is incompressible near zero velocity. Let $z = x + iy$ and z_T be the value of z at the trailing edge and $\varepsilon\pi$ the half-angle there. Map a neighborhood of the trailing edge onto a vertical half disc, $\mathscr{Re}\,\zeta > 0, |\zeta| < 1$ in such a way that the trailing edge goes into the origin, and a part of the profile into the imaginary axis. Then $\zeta \sim \text{const.}\ (z - z_T)^{1/(1-\varepsilon)}$. As a function of $\zeta = \xi + i\eta$, $\Phi + i\Psi$ is analytic at $\zeta = 0$ and $\Phi + i\Psi = a_0^1(\tau)\zeta(1 + a_1^1(\tau)\zeta + \cdots)$. It may be easily justified that we may differentiate with respect to τ and obtain $\omega + i\omega^* = a_{0\tau}(0)\zeta(1 + O(\zeta))$. Differentiating with respect to z we obtain

$$\omega_x - i\omega_y \sim \text{const.}\ (z - z_T)^{\epsilon/(1-\epsilon)} \tag{34}$$

and from the expression for $\Phi + i\Psi$ at $\tau = 0$,

$$\varphi_x - i\varphi_y = u - iv \sim \text{const.}\ (z - z_T)^{\epsilon/(1-\epsilon)}. \tag{35}$$

Furthermore differentiating (35) we find that x_u, x_v, y_u, y_v are $O\big((u - iv)^{1/(\varepsilon-2)}\big)$. Thus ω_u and ω_v are $O\big((u - iv)^{(1-\varepsilon)/\varepsilon}\big)$. By (20) and (22) it follows that as $\sigma \to \infty$, (33) holds for Π_T since $\varepsilon \leqq \frac{1}{2}$. Similarly we can establish (33) for Π_N.

The behavior of ω_θ and ω_σ at Π_∞ is derived from (13*) and (22),

$$\omega_\theta,\ \omega_\sigma \text{ are } o\big(|\theta + i(\sigma - \sigma_\infty)|^{-1/2}\big) \text{ at } \Pi_\infty\,. \tag{36}$$

Problem II has been transformed into Problem III in the θ, σ-plane; a function $\omega(\theta, \sigma)$ is a solution of Problem III if it satisfies (25), (26), (31), (32), (33), (36) in the domain $\Omega + \Omega^*$. Theorem II is then equivalent to showing that any solution of Problem III is constant in $\Omega + \Omega^*$. This follows from Theorems III and IV.

THEOREM III. *The solution of Problem III is constant in Ω.*

THEOREM IV. *Suppose ω satisfies* (25), (31) and (32) *in a domain R bounded by four characteristics $\gamma_1, \gamma_2, \gamma_3, \gamma_4$* where γ_1 and γ_2 *correspond to greater values of σ. If ω is prescribed on γ_1, γ_2 then ω is uniquely determined in R. Or equivalently if $\omega = 0$ on $\gamma_1 + \gamma_2$, $\omega \equiv 0$ in R.*

Theorem IV which is a slight generalization of a standard theorem is proved in Appendix 2. The domain R is taken to be the domain bounded by γ_1, γ_2 and the characteristics through their end points. Then $\omega \equiv 0$ in $\Omega + R \supset \Omega + \Omega^*$.

3. Proof of Theorem III

Theorem III is proved by the ABC-method of Friedrichs. Let ω be a solution of Problem III and consider the integral

$$I_\delta = \iint_{\Omega_\delta} (B\omega_\theta + C\omega_\sigma)(K\omega_{\theta\theta} + \omega_{\sigma\sigma})\, d\theta d\sigma = 0 \tag{37}$$

where Ω_δ is a subdomain of Ω and B and C are functions of θ and σ to be chosen later. The domain Ω_δ is bounded by the curve $\mathscr{B}_\delta$ which consists of the two characteristics γ_1 and γ_2 and the boundary curves $\Pi - \Sigma$ and Λ, except in the neighborhood of Π_0, Π_1, Π_2 and Π_∞ where it consists of four curves $R_{0\delta}$, $R_{1\delta}$, $R_{2\delta}$ and $R_{3\delta}$ which exclude Π_0, Π_1, Π_2 and Π_∞ and whose length shrinks to zero as $\delta \to 0$, see Figure 3. $R_{0\delta}$, $R_{1\delta}$, $R_{2\delta}$ and $R_{3\delta}$ will be

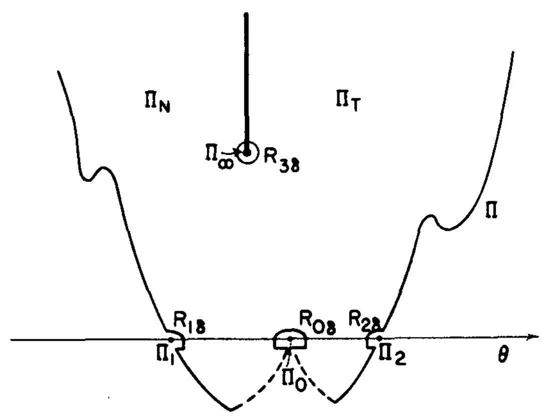

Figure 3.

chosen more specifically later. We denote the portion of $\mathscr{B}_\delta$ which coincides with $\Pi - \Sigma$, γ_1, γ_2, by Π_δ, $\gamma_{1\delta}$, $\gamma_{2\delta}$, respectively.

We shall show how to choose B and C in such a way that unless ω is constant

$$\lim_{\delta \to 0} I_\delta > 0.$$

This contradiction will prove the theorem.

First B and C are to be chosen so that we may integrate (37) by parts and obtain

$$\begin{aligned} I_\delta = \iint_{\Omega_\delta} \Big\{ \frac{1}{2} \omega_\theta^2 (-KB_\theta + (KC)_\sigma) - \omega_\theta \omega_\sigma (B_\sigma + KC_\theta) \\ + \frac{1}{2} \omega_\sigma^2 (B_\theta - C_\sigma) \Big\} d\theta d\sigma \\ + \oint_{\mathscr{B}_\delta} \frac{B}{2} (K\omega_\theta^2 d\sigma - 2\omega_\theta \omega_\sigma d\theta - \omega_\sigma^2 d\sigma) \\ + \frac{C}{2} (K\omega_\theta^2 d\theta + 2K\omega_\theta\omega_\sigma d\sigma - \omega_\sigma^2 d\theta) \end{aligned} \tag{38}$$

or

$$I_\delta = I_{1\delta} + I_{2\delta} + I_{3\delta} + I_{4\delta} \tag{39}$$

where $I_{1\delta}$, $I_{2\delta}$, $I_{3\delta}$, $I_{4\delta}$ are the area and line integrals over the elliptic and hyperbolic regions respectively. Thus,

$$\begin{aligned} I_{1\delta} &= \iint_{\Omega_\delta, \sigma \geqq 0} \Big\{ \frac{1}{2} \omega_\theta^2 (-KB_\theta + (KC)_\sigma) - \omega_\theta\omega_\sigma (B_\sigma + KC_\theta) + \frac{1}{2} \omega_\sigma^2 (B_\theta - C_\sigma) \Big\} d\theta d\sigma, \\ I_{2\delta} &= \oint_{\mathscr{B}_\delta, \sigma \geqq 0} \frac{B}{2} (K\omega_\theta^2 d\sigma - 2\omega_\theta\omega_\sigma d\theta - \omega_\sigma^2 d\sigma) + \frac{C}{2} (K\omega_\theta^2 d\theta + 2K\omega_\theta\omega_\sigma d\sigma - \omega_\sigma^2 d\theta). \end{aligned} \tag{40}$$

By (26) we replace $\omega_\sigma d\theta$ by $K\omega_\theta d\sigma$ in the first parenthesis and $K\omega_\theta d\sigma$ by $\omega_\sigma d\theta$ in the second, and obtain

$$\begin{aligned} I_{2\delta} = \oint_{\Pi_\delta, \sigma \geqq 0} \frac{1}{2} (K\omega_\theta^2 + \omega_\sigma^2)(Cd\theta - Bd\sigma) + \sum_{i=0}^{3} \oint_{R_{i\delta}} \frac{B}{2} (K\omega_\theta^2 d\sigma - 2\omega_\theta\omega_\sigma d\theta - \omega_\sigma^2 d\sigma) \\ + \frac{C}{2} (K\omega_\theta^2 d\theta + 2K\omega_\theta\omega_\sigma d\sigma - \omega_\sigma^2 d\theta). \end{aligned} \tag{41}$$

The integrals in the hyperbolic region are,

$$\begin{aligned} I_{3\delta} &= \iint_{\Omega_\delta, \sigma \leqq 0} \Big\{ \frac{1}{2} \omega_\theta^2 (-KB_\theta + (KC)_\sigma) - \omega_\theta\omega_\sigma (B_\sigma + KC_\theta) + \frac{1}{2} \omega_\sigma^2 (B_\theta - C_\sigma) \Big\} d\theta d\sigma, \\ I_{4\delta} &= \oint_{\mathscr{B}_\delta, \sigma \leqq 0} \frac{B}{2} (K\omega_\theta^2 d\sigma - 2\omega_\theta\omega_\sigma d\theta - \omega_\sigma^2 d\sigma) + \frac{C}{2} (K\omega_\theta^2 d\theta + 2K\omega_\theta\omega_\sigma d\sigma - \omega_\sigma^2 d\theta) \end{aligned} \tag{42}$$

which, by (26) and (30), may be written as

$$I_{4\delta}=\int\limits_{\gamma_{1\delta}}\frac{1}{2}(-Bd\sigma-Cd\theta)(\omega_\sigma+\sqrt{-K}\omega_\theta)^2+\int\limits_{\gamma_{2\delta}}\frac{1}{2}(-Bd\sigma-Cd\theta)(\omega_\sigma-\sqrt{-K}\omega_\theta)^2$$

$$(43)\quad +\int\limits_{\Pi_\delta,\sigma\leqq 0}\frac{K\omega_\theta^2}{2}\left(1+K\left(\frac{d\sigma}{d\theta}\right)^2\right)(Cd\theta-Bd\sigma)+\int\limits_{R_{0\delta}+R_{1\delta}+R_{2\delta},\,\sigma\leqq 0}\frac{B}{2}(K\omega_\theta^2 d\sigma$$
$$-2\omega_\theta\omega_\sigma d\theta-\omega_\sigma^2 d\theta)+\frac{C}{2}(K\omega_\theta^2 d\theta+2K\omega_\theta\omega_\sigma d\sigma-\omega_\sigma^2 d\theta).$$

We note that $1+K\left(\frac{d\sigma}{d\theta}\right)^2\geqq 0$ by (29).

By (31), the integrals will exist for all $\delta\neq 0$ if B and C are continuous in the closure of Ω_δ and are bounded at Π_N and Π_T by (33).

We shall show that functions B and C exist such that $I_{1\delta}$ and $I_{3\delta}$ are positive and bounded waay from zero uniformly in δ and such that $\underline{\lim}_{\delta\to 0} I_{2\delta}$ and $\underline{\lim}_{\delta\to 0} I_{4\delta}$ are non-negative. It follows then that $\omega_\theta\equiv\omega_\sigma\equiv 0$ in Ω. This will prove Theorem III.

The integrals $I_{1\delta}$ and $I_{2\delta}$ are positive definite and $\underline{\lim}_{\delta\to 0} I_{2\delta}$, $\underline{\lim}_{\delta\to 0} I_{1\delta}$ are non-negative if

$$(44)\quad \begin{aligned} -KB_\theta+(KC)_\sigma &> 0 && \text{in } \Omega,\ \sigma\neq 0,\\ B_\sigma+KC_\theta &= 0, &&\\ B_\theta-C_\sigma &> 0 && \text{in } \Omega,\ \sigma\neq 0,\\ K(Cd\theta-Bd\sigma) &\geqq 0 && \text{on } \Pi_\delta+\Lambda,\\ -Cd\theta-Bd\sigma &\geqq 0 && \text{on } \gamma_1 \text{ and } \gamma_2, \end{aligned}$$

and if

$$(45)\quad \underline{\lim}_{\delta\to 0} I^+_{R_{i\delta}}=\underline{\lim}_{\delta\to 0} I^-_{R_{i\delta}}=0$$

where $I^+_{R_{i\delta}}$, $I^-_{R_{i\delta}}$, are the line integrals along $R_{i\delta}$ for $\sigma\geqq 0$, $i=0,1,2,3,$ and $\sigma\leqq 0$ respectively, $i=0,1,.2.$

In the lower half plane, $\sigma\leqq 0$, we set

$$(46)\quad \begin{aligned} B&=B(\theta,0), \qquad \sigma<0,\\ C&=0. \end{aligned}$$

Then conditions (44) for $\sigma<0$ will be satisfied if

$$(47)\quad \begin{aligned} B_\theta(\theta,0)&>0 && \text{on } \theta_1\leqq\theta\leqq\theta_2,\\ B(\theta,0)&\geqq 0 && \text{for } \theta_0\leqq\theta\leqq\theta_2,\\ B(\theta,0)&\leqq 0 && \text{for } \theta_1\leqq\theta\leqq\theta_0, \end{aligned}$$

where $\theta_i=\theta(\Pi_i)$, $i=0,1,2.$

For the purpose of constructing B and C for $\sigma \geqq 0$ we introduce an auxiliary plane[4] into which we map Ω, $\sigma \geqq 0$. We set

$$\mu = \int_0^\sigma \sqrt{K(\sigma)}\, d\sigma. \tag{48}$$

Then the upper half of the θ, σ-plane is mapped smoothly into the upper half of the λ-plane where $\lambda = \theta + i\mu$. Let $\Pi_0, \Pi_1, \Pi_2, \Pi_T, \Pi_N, \Pi_\infty$ be mapped into $\Pi_0^*, \Pi_1^*, \Pi_2^*, \Pi_T^*, \Pi_N^*, \Pi_\infty^*$. Let $R_{i\delta}, \Pi_\delta, \Omega_\delta, \Lambda$ and $\Pi - \Sigma$ for $\sigma \geqq 0$ be mapped into $R_{i\delta}^*, \Pi_\delta, \Omega_\delta, \Pi^*$. Note that since on $\Pi - \Sigma$, $d\theta > 0$ for $\theta \neq 0$ we have

$$d\theta > 0 \qquad \text{on } \Pi^*, \qquad \theta \neq 0. \tag{49}$$

Using Privaloff's theorem on conjugate functions it is easy to show that there exists a function $f(\lambda) = f_1(\theta, \mu) + i f_2(\theta, \mu)$ which is analytic in $\lambda = \theta + i\mu$ in the interior of Ω^*, continuous in the closure of Ω^* except at $\Pi_0^*, \Pi_1^*, \Pi_2^*, \Pi_T^*, \Pi_N^*$, and satisfies the boundary conditions:

$$\begin{aligned} f_2 &= 0 \qquad \text{on } \Lambda^*, \\ f_2 &= jg \qquad \text{on } \Pi^* \end{aligned} \tag{50}$$

where $\cos j = d\theta/ds$, $\sin j = d\mu/ds$ on Π^* and by (49) we may take $-\pi/2 \leqq j \leqq \pi/2$, g is a given very smooth, say infinitely differentiable, function satisfying

$$\begin{aligned} &0 \leqq g \leqq 1, \\ &g = 0 \text{ near } \Pi_T^*, \Pi_N^*, \Pi_1^*, \Pi_2^* \text{ where } \theta d\mu > 0. \end{aligned} \tag{51}$$

From the assumption that the second derivatives of φ are Hölder continuous it can be easily shown that jg is Hölder continuous. On $\mu = 0$,

$$\begin{aligned} f_2 &= +\frac{\pi}{2} \qquad \text{for } \theta_1 \leqq \theta \leqq \theta_0, \\ f_2 &= -\frac{\pi}{2} \qquad \text{for } \theta_0 \leqq \theta \leqq \theta_2. \end{aligned} \tag{52}$$

At Π_1^* and Π_2^* where $\lambda = \theta_i$, $i = 1, 2$, it can be shown that

$$f = -\tfrac{1}{2} \log (\lambda - \theta_i) + \text{continuous function.} \tag{52}$$

At Π_0^* where $\lambda = \theta_0$,

$$f = \log(\lambda - \theta_0) + \text{continuous function.} \tag{53}$$

At Π_T^*, Π_N^* were $\mu \to \infty$,

$$f \text{ is bounded.} \tag{54}$$

[4] In the variables (θ, μ), equation (25) takes on the normal form.

The functions B and C are chosen for $\sigma \geqq 0$ as

$$\sqrt{K}C - iB = \exp \{f(\lambda(\theta, \mu))\}. \tag{55}$$

We must show now (a) that B and C are continuous at $\sigma = 0$ and all the integrals $I_{1\delta}$, $I_{2\delta}$, $I_{3\delta}$, $I_{4\delta}$ exist and (b) that the conditions (44) and (45) are satisfied.

From the properties of $f(\lambda)$ we see that e^f is a regular analytic function of the complex variable λ in Ω^* with a square root singularity at Π_1^* and Π_2^*, and vanishes to first order at Π_0^* by (52) and (53). As $\mu \to \infty$ at Π_T^* and Π_N^*, e^f is bounded by (54). Also $\mathcal{R}e\, e^f = 0$ on $\mathcal{I}m\, \lambda = 0$.

By (50), (51) and (52) we have $|f_2| < \max\, (\pi/2, |j|)$ on the boundary and by (49), $|j| < \pi/2$. Therefore $|f_2| < \pi/2$ and $\sqrt{K}C > 0$ almost everywhere in Ω^* since by (50), (51) and (52) it is not a constant. Furthermore by (52), $\sqrt{K}C = 0$ on $\mu = 0$ and therefore since $\sqrt{K}C - iB$ is analytic up to $\mu = 0$, by the reflection principle, we have

$$\begin{aligned} &\sqrt{K}C = 0 && \text{on } \mu = 0, \\ &\sqrt{K}C > 0 && \text{in } \Omega^*, \\ &(\sqrt{K}C)_\mu > 0 && \text{on } \mu = 0. \end{aligned} \tag{56}$$

We next check (a) and (b).

(a) By (56) $\sqrt{K}C$ vanishes at least like μ on $\mu = 0$. Therefore C vanishes like σ on $\sigma = 0$ by (48) and (24). By (47) then B and C are continuous on $\sigma = 0$ in Ω_δ.

Plainly the integrals $I_{1\delta}$, $I_{2\delta}$, $I_{3\delta}$, $I_{4\delta}$ converge absolutely for all $\delta \neq 0$ since B and C are bounded and continuous in Ω_δ, $\delta \neq 0$.

(b) To check (45) we must fix $R_{i\delta}$. For $\sigma \geqq 0$, $R_{i\delta}$, $i = 0, 1, 2$, is the image in the θ, σ-plane of a semi-circle $R_{i\delta}^*$ of radius δ in the θ, μ-plane. For $\sigma \leqq 0$, $R_{i\delta}$ consists of the segments cut out by the lines $\sigma = -\delta$, $|\theta - \theta_i| = \delta$, see Figure 3. $R_{3\delta}$ is a circle of radius δ.

First we consider the line integral at Π_∞,

$$I_{R_{3\delta}}^+ = \oint \frac{B}{2}\,(K\omega_\theta^2 d\sigma - 2\omega_\theta\omega_\sigma d\theta - \omega_\sigma^2 d\sigma) + \frac{C}{2}\,(K\omega_\theta^2 d\theta + 2K\omega_\theta\omega_\sigma d\sigma - \omega_\sigma^2 d\theta).$$

By (36), ω_θ and ω_σ are $o(\delta^{-\frac{1}{2}})$. The coefficients B and C are bounded. Therefore $\lim\limits_{\delta \to 0} I_{R_{3\delta}} = 0$.

For $I_{R_{i\delta}}^+$, $i = 0, 1, 2$, we use the θ, μ-plane:

$$
\begin{aligned}
I^+_{R_{i\delta}} &= \oint_{R_{i\delta},\,\sigma\geqq 0} \frac{B}{2}\,(K\omega_\theta^2 d\sigma - 2\omega_\theta\omega_\sigma d\theta - \omega_\sigma^2 d\sigma) \\
&\quad + \frac{C}{2}\,(K\omega_\theta^2 d\theta + 2\omega_\theta\omega_\sigma d\sigma - \omega_\sigma^2 d\theta) \\
&= \oint_{R^*_{i\delta}} \frac{B}{2}\left(\sqrt{K}\omega_\theta^2 d\mu - 2\omega_\theta\omega_\sigma d\theta = \omega_\sigma^2 \frac{d\mu}{\sqrt{K}}\right) \\
&\quad + \frac{\sqrt{K}C}{2}\left(\sqrt{K}\omega_\theta^2 d\theta + 2\omega_\theta\omega_\sigma d\mu - \frac{\omega_\sigma^2}{\sqrt{K}}\, d\theta\right).
\end{aligned}
$$

Now from (52), (53) and (55) we have

$$\sqrt{K}C - iB = O(\lambda - \theta_i)^{\bar{\varkappa}_i}$$

where $\bar{\varkappa}_i = 1$, $\theta_i = \theta_0$ at Π_0 and $\bar{\varkappa}_i = -\frac{1}{2}$, $\theta_i = \theta_1$, θ_2 at Π_1 and Π_2.

From (24), $dK/d\sigma > 0$, we see that $K \sim K'(0)\sigma$ and by (48), $\mu \sim \frac{2}{3}(K'(0))^{1/2}\sigma^{3/2}$, $K \sim (\frac{3}{2})^{2/3}(K'(0))^{2/3}\mu^{2/3}$. Hence since ω_θ, ω_σ are, by (32), $o(((\theta-\theta_i)^2+\sigma^2)^{\varkappa_i/2})$ they are $o(\delta^{2\varkappa_i/3})$ on $R^*_{i\delta}$. We set $\tan\nu = \mu/(\theta-\theta_i)$ on $R^*_{i\delta}$, then $\sqrt{K} = (\frac{3}{2})^{1/3}(K'(0))^{1/3}\delta^{1/3}\sin^{1/3}\nu$ on $R^*_{i\delta}$ and thus

$$\left|I^+_{R_{i\delta}}\right| = o(\delta^{\frac{4}{3}\varkappa_i+\bar{\varkappa}_i+\frac{2}{3}})\left[\int_0^\pi \sin^{-\frac{1}{3}}\nu\, d\nu + \int_0^\pi d\nu\right] = o(\delta^{\frac{4}{3}\varkappa_i+\bar{\varkappa}_i+\frac{2}{3}}).$$

Substituting the values of $\bar{\varkappa}_i$ from above and $\varkappa_i$ from (32)

$$I^+_{R_{i\delta}} = o(1).$$

Hence

$$\lim_{\delta\to 0} I^+_{R_{i\delta}} = 0.$$

For the integrals in the hyperbolic region we have, using (46),

$$I^-_{R_{i\delta}} = \int_{R_{i\delta},\,\sigma\leqq 0} B(\theta, 0)(\tfrac{1}{2}K\omega_\theta^2 d\sigma - \omega_\theta\omega_\sigma d\theta - \tfrac{1}{2}\omega_\sigma^2 d\theta).$$

By (52), (53) and (55), $|B(\theta, 0)| = O((\theta-\theta_i)^{\bar{\varkappa}_i})$ and $|B(\theta, 0)| \sim$ const. $\delta^{\bar{\varkappa}_i}$ on $|\theta - \theta_i| = \delta$. By (32), $|\omega_\theta|$ and $|\omega_\sigma|$ are $o(\delta^{-\varkappa_i})$ on all of $R^-_{i\delta}$. Hence

$$I^-_{R_{i\delta}} = o(\delta^{-2\varkappa_i+\bar{\varkappa}_i+1}) + o(\delta^{-2\varkappa_i})\int_0^\delta |\theta_i - \theta|^{\bar{\varkappa}_i}\, d\theta = o(\delta^{-2\varkappa_i+\bar{\varkappa}_i+1}),$$

and by (32), $\lim\limits_{\delta\to 0} R^-_{i\delta} = 0$. Condition (45) has been proved.

It remains only to establish the inequalities (44) and (47). We transform (44) and (47) to the θ, μ-plane and show that

$$-K(B_\theta - (\sqrt{K}C)_\mu) + \frac{1}{2}\frac{dK}{d\mu}\sqrt{K}C > 0 \qquad \text{in } \Omega^*, \tag{57}$$

$$B_\mu + (\sqrt{K}C)_\theta = 0, \tag{58}$$

$$B_\theta - (\sqrt{K}C)_\mu + \frac{1}{2}\frac{dK}{d\mu}\frac{C}{\sqrt{K}} > 0 \qquad \text{in } \Omega^*, \tag{59}$$

$$\sqrt{K}C\,d\theta - Bd\mu \geqq 0 \qquad \text{on } \Pi,\ \Lambda, \tag{60}$$

$$B_\theta > 0 \qquad \text{on } \mu = 0, \tag{61}$$

$$\begin{aligned} B &\leqq 0 \qquad \text{on } \mu = 0, \quad \theta_1 \leqq \theta \leqq \theta_0, \\ B &\geqq 0 \qquad \text{on } \mu = 0, \quad \theta_0 \leqq \theta \leqq \theta_2. \end{aligned} \tag{62}$$

Since $\sqrt{K}C - iB$ is analytic in $\lambda = \theta + i\mu$, (57), (58) and (59) hold by virtue of the Cauchy-Riemann equations, (56) and (24). Using (55) and (50), on Π,

$$\begin{aligned} \sqrt{K}C\,d\theta - Bd\mu &= \exp\{\mathscr{R}e\, f\}\cos(j - \mathscr{I}m\, f) \\ &= \exp\{\mathscr{R}e\, f\}\cos(j(1-g)). \end{aligned}$$

Now $|j(1-g)| \leqq |j|$ by (51) and by (49), $-\pi/2 \leqq j \leqq \pi/2$, so that $\cos j(1-g) > 0$. On Λ, $d\theta = 0$, $B = 0$. Hence (60) holds.

(61) follows from (56) since $(\sqrt{K}C)_\mu = B_\theta$. Finally by (55), $B = -\exp\{\mathscr{R}e\, f\}\sin \mathscr{I}m\, f$ and condition (62) follows from (52).

The proof of Theorem III is now complete.

Appendix 1

Behavior at Infinity and at the Nose and Trailing Edge

In this appendix the references labelled B are to Bers [11].

At infinity. Proof of Lemma 1. First we shall need the asymptotic behavior of the unperturbed velocity at infinity. We use the notations of B Lemma 8a. The distorted velocity w^* for the flow $\mathscr{F}_0$ is an analytic function of $\zeta = \xi + i\eta$ where ζ is a mapping of a neighborhood of $z = \infty$ onto a neighborhood of $\zeta = \infty$, conformal to the flow metric. The distorted velocity w^* is defined in the subsonic region in terms of the velocity $w = \varphi_x - i\varphi_y$, see B (7.7). For $\zeta \to \infty$ it follows that

$$w^* = w^*_\infty + \frac{w^*_{1\infty}}{\zeta} + \frac{w^*_{2\infty}}{\zeta^2} + \cdots. \tag{A1}$$

We may assume the mappings $z \rightleftarrows \zeta$ are symmetric with respect to the horizontal axes. Then $w^*_{1\infty}$, $w^*_{2\infty}$, $\cdots$ are all real.

As $w \to q_\infty = w_\infty > 0$ in the notation of Bers (11), we find by expanding B (7.7), for $|z| \to \infty$

(A2) $$|w^*| - |w^*_\infty| = A(|w| - |w_\infty|)(1 + o(1))$$

where A is a positive constant. Hence from (A1)

(A3) $$|w| - |w_\infty| = \frac{(1 + o(1))}{A}\left(\left|w^*_\infty + \frac{w^*_{1\infty}}{\zeta} + \frac{w^*_{2\infty}}{\zeta^2} + O(\zeta^{-3})\right| - |w^*_\infty|\right).$$

Expanding $\arg w = \arg w^*$ in powers of ζ from (A1) and combining with (A3) we obtain

(A4) $$w - w_\infty = w^*_{1\infty}\left(\frac{(1 + o(1))}{A}\mathscr{Re}\,\zeta^{-1} + \frac{w_\infty}{w^*_\infty}\mathscr{Im}\,\zeta^{-1}\right) + O(\zeta^{-2}).$$

The mapping from the physical plane into the ζ-plane is conformal with respect to the flow metric:

(A5) $$d\xi^2 + d\eta^2 = (c^2 - u^2)\,dx^2 + 2uv\,dx\,dy + (c^2 - v^2)\,dy^2.$$

Therefore as $\zeta \to \infty$, $u \to w_\infty$, $v \to 0$ we may choose the mapping so that

(A6) $$y_\xi \to \frac{1}{\sqrt{1 - M^2_\infty}},\quad y_\eta \to 1,\quad x_\eta \to x_\xi \to 0 \text{ as } z \to \infty$$

and thus

(A7) $$\zeta = (x + i\sqrt{1 - M^2_\infty}\,y)(1 + o(1)).$$

Substituting (A7) in (A4) we have

(A8) $$w - w_\infty = w^*_{1\infty}\left(\frac{1}{A}\mathscr{Re}(x + i\sqrt{1 - M^2_\infty}\,y)^{-1} + \frac{w_\infty}{w^*_\infty}\mathscr{Im}(x + i\sqrt{1 - M^2_\infty}\,y)^{-1}\right.$$
$$\left. + o\left((x + i\sqrt{1 - M^2_\infty}\,y)^{-1}\right)\right) + O\left((x + i\sqrt{1 - M^2_\infty}\,y)^{-2}\right)$$

Now

$$\psi = \int_{z_T}^{z} \varrho v\,dx - \varrho u\,dy = -\mathscr{Im}\int_{z_T}^{z} \varrho w\,dz$$

is single-valued since $\psi = 0$ on the profile P_0. Thus $\mathscr{Im}\oint \varrho w\,dz = 0$ where the path of integration may be taken as the ellipse $x^2 + (1 - M^2_\infty)y^2 = R^2$. For $R \to \infty$, we find by direct computation that

$w^*_{1\infty}\left(\frac{w_\infty}{w^*_\infty} + \frac{1}{A\sqrt{1 - M^2_\infty}}\right) = 0$ and therefore $w^*_{1\infty} = 0$. Thus[5] from (A8),

[5] cf. [12].

(A9) $$w - w_\infty = O((x + i\sqrt{1 - M_\infty^2 y})^{-2})$$

and in (A1),

(A10) $$w^* = w_\infty^* + \frac{w_{2\infty}^*}{\zeta^2} + O(\zeta^{-3}).$$

Secondly, we must derive estimates for x_u, x_v, y_u, y_v at ∞ which we shall need to estimate $\omega_u - i\omega_v$, we differentiate (A10) with respect to ζ and find $d\zeta/dw^*$; differentiate B(7.7) with respect to u and v and use (A6). This shows that x_u, x_v, y_u, y_v are $O(|w - q_\infty|^{-3/2})$ or by (20), (22) and (48), x_θ, x_μ, y_θ, y_μ are $O(|\lambda - \lambda_\infty|^{-3/2})$ where

$$\lambda = \theta + i\mu, \qquad \lambda_\infty = i\mu(q_\infty).$$

Since $\omega_x - i\omega_y \to 0$ it follows that

(A11) $$\omega_\theta - i\omega_\mu = O(|\lambda|^{-3/2}).$$

Finally, to derive (11), we use the fact that $\omega_\theta - i\omega_\mu$ is a pseudo-analytic function of the first kind. For, since ω and ω^* satisfy (2), they satisfy (23) and therefore ω satisfies

(A12) $$\omega_{\theta\theta} + \omega_{\mu\mu} + \frac{dK}{Kd\mu}\omega_\mu = 0$$

where $\mu = \int_0^\sigma \sqrt{K(\sigma)}\, d\sigma$, (48). It follows by the similarity principle, see [13], that in the neighborhood of $(0, \mu_\infty)$

(A13) $$\omega_\theta - i\omega_\mu = \exp\{s(\theta, \mu)\}\, W(\lambda - \lambda_\infty).$$

Here s is Hölder continuous, real on $\theta = 0$ and vanishes at $(0, \mu_\infty)$, and W is analytic. Since ω^* satisfies (5) on $y = 0$ we have by (26), $\omega_\theta = 0$ on $\theta = 0$, $\mu \geqq \mu_\infty$. Introducing $\lambda^* = (-i(\lambda - \lambda_\infty))^{1/2} = \theta^* + i\mu^*$ we see that $\mathcal{R}e\, W$ vanishes on $\mathcal{I}m\, \lambda^* = 0$ and by reflecting about $\mathcal{I}m\, \lambda^* = 0$ that W is a singlevalued function of λ^*, $0 < |\lambda^*| < |\lambda_0^*|$. Therefore by (A11)

(A14) $$\omega_\theta - i\omega_\mu = e^s \sum_{-2}^{\infty} a_n \lambda^{*n}.$$

Since s is continuous and vanishes at $(0, \mu_\infty)$,

(A15) $$\omega_\theta - i\omega_\mu = a_{-2}\lambda^{*-2}(1 + o(1)) + a_{-1}\lambda^{*-1}(1 + o(1)).$$

From (23) and (26) we have

$$\oint d\omega^* = -\mathcal{I}m \oint \sqrt{K}(\omega_\theta - i\omega_\mu)(d\theta + id\mu) = 0$$

where the path of integration is a circle of radius δ about $(0, \mu_\infty)$.

Letting $\delta \to 0$ we obtain by (A15), $\mathscr{I}m\, 2\pi i a_{-2} = 0$. By (26) and $\mathscr{I}m\, s = 0$ on $\theta = 0$ we have $\mathscr{I}m\, a_{-2} = 0$. Hence $a_{-2} = 0$.

Therefore

$$\omega_\theta - i\omega_\mu = a_{-1}\lambda^{*-1}(1 + o(1)). \tag{A16}$$

By (48), (22), (20) and (9),

$$\begin{aligned} \omega_u &= \omega_\mu q_\infty^{-1}(1 - M_\infty^2)^{1/2}(1 + o(1)) + \omega_\theta o(1), \\ \omega_v &= \omega_\theta q_\infty^{-1}(1 + o(1)) + \omega_\mu(o(1)). \end{aligned} \tag{A17}$$

Equations (A17) yield (11).

At the Nose and Trailing Edge. Derivation of (33). For a compressible flow $|\varphi_x - i\varphi_y| = |w| \sim \text{const}\, |z - z_T|^{\varepsilon/(1-\varepsilon)}$ where $\varepsilon\Pi$ is the half angle of the trailing edge, $z = x + iy$ and z_T is the value of z at the trailing edge. This relation is determined in B p. 478.

To estimate x_u, x_v, y_u, y_v we use the method of B §8. We can derive a representation of the form B(8.7) for the distorted velocity

$$w^*(\zeta) = e^{G(\zeta)}\left(1 - \frac{1}{\zeta}\right)^{\varepsilon}.$$

Here ζ is again a mapping conformal to the flow metric, say, of the whole subsonic domain onto the exterior of the unit circle. The point $\zeta = 1$ corresponds to the trailing edge and the function $G(\zeta)$ is analytic and Hölder continuous at $\zeta = 1$. The last remark implies that the derivatives of $G(\zeta)$ are $O(|\zeta - 1|^{1-\alpha})$ for some $\alpha > 0$. Hence by differentiating $w^*(\zeta)$ we estimate $dw^*/d\zeta$ or $d\zeta/dw^*$. From B (7.7) differentiated we find dw^*/dw. From the flow metric we determine x_ξ, x_η, y_ξ, y_η. We find that x_u, x_v, y_u, y_v are $O(|w|^{1/(\varepsilon-2)})$.

Since $\omega_x - i\omega_y \to 0$ at the stagnation points we have $\omega_u - i\omega_v = o(w^{\frac{1}{\varepsilon}-2})$ or, by (13) and (15), ω_θ and ω_σ are $o\left(\exp\left\{i\left(\frac{1}{\varepsilon} - 1\right)(\theta + i\sigma)\right\}\right)$.

Since $\varepsilon < \frac{1}{2}$ it follows that $\omega_\theta e^\sigma$, $\omega_\sigma e^\sigma$ are bounded, (33).

The leading edge may be treated in the same way.

Appendix 2. Proof of Theorem IV

Let ω vanish on $\gamma_1 + \gamma_2$ and ω^* vanish at some point on $\gamma_1 + \gamma_2$. We again apply the ABC-method but to the function $W = \int_0^{\theta, \sigma} \omega^* d\sigma + \omega\, d\theta$. It is not difficult to show by (23) that W is single-valued, vanishes on γ_1

and γ_2 and satisfies (25). Let the intersection of γ_1 and γ_2 be at (θ_0, σ_0) and that of γ_3 and γ_4 be at (θ_0, σ_1) where $\sigma_0, \sigma_1 \leqq 0$. From (25), as in (38), we have for any subdomain R_δ of R, bounded by G_δ

$$\begin{aligned} 0 &= \iint_{R_\delta} CW_\sigma(KW_{\theta\theta} + W_{\sigma\sigma})d\theta d\sigma \\ &= \oint_{G_\delta} \frac{C}{2} KW_\theta^2 d\theta + 2W_\theta W_\sigma d\sigma - W_\sigma^2\, d\theta) \\ &\quad + \iint_{R_\delta} \{\tfrac{1}{2}W_\theta^2(KC)_\sigma - W_\theta W_\sigma KC_\theta - \tfrac{1}{2}W_\sigma^2 C_\sigma\}\, d\theta d\sigma. \end{aligned} \tag{A18}$$

We take R_δ as the domain bounded by $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and the segment S_δ cut out by $\sigma = \sigma_0 - \delta$. Then substituting $W_\sigma = \omega^*$, $W_\theta = \omega$, yields

$$\begin{aligned} 0 &= \oint_{\gamma_1+\gamma_2} -\frac{Cd\theta}{2}(\sqrt{-K}\omega + \omega^*)^2 + \oint_{\gamma_3+\gamma_4} -\frac{Cd\theta}{2}(\sqrt{-K}\omega - \omega^*)^2 \\ &\quad + \int_{S_\delta} \frac{C}{2}(K\omega^2 d\theta + 2K\omega\omega^* d\sigma - \omega^{*2} d\theta) \\ &\quad + \iint_{R_\delta} \{\tfrac{1}{2}\omega^2(KC)_\sigma - \omega\omega^* KC_\theta - \tfrac{1}{2}\omega^{*2}C_\sigma\} d\theta d\sigma \end{aligned} \tag{A19}$$

where γ_1 and γ_3 have positive slope $\sqrt{-K}$ and γ_2 and γ_4 have negative slope $-\sqrt{-K}$, see (30).

We take $C = 1/K$. The integrals along S_δ are arbitrarily small as $\delta \to 0$. For, $d\sigma = 0$, ω and ω^* are easily shown to be bounded by (32), and thus the integral is bounded,

$$\left| \oint_{S_\delta} \frac{d\theta}{2K}(K\omega^2 - \omega^{*2}) \right| < \frac{\text{const.}}{K(\delta)} \int_{S_\delta} d\theta.$$

But the length of S_δ is

$$\int_{S_\delta} d\theta = O\left(2 \int_0^\delta \sqrt{-K(\sigma)} \cdot d\sigma\right) = O(\delta^{3/2})$$

since $dK/d\sigma \neq 0$. Hence the integral along S_δ is $O(\delta^{1/2})$ and has limit zero.

On γ_1 and γ_2 the line integrals vanish since $W = 0$. On γ_3 and γ_4 the line integrals are non-negative since $C < 0$, $d\theta > 0$. The area integral reduces to

$$\iint_{R_\delta} + \frac{K'}{2K^2}\omega^{*2}\, d\theta d\sigma \leqq 0.$$

But it is positive unless $\omega^* \equiv 0$ in R_θ or in the limit $\omega^* \equiv 0$, $\omega \equiv 0$ in R.

Appendix 3. Flow with Weak Shocks

We shall show that we cannot lift the continuity restrictions (14) on ω_x or ω_y in the supersonic region of a continuous flow and obtain a perturbation solution representing a flow with a weak shock.

Let us suppose that the perturbation velocities ω_x, ω_y have some discontinuities in the supersonic region, that is for $\sigma < 0$. For a weak shock we may neglect changes of entropy and vorticity, see [5], §117, and derive the shock conditions by integrating the equations of motion. This yields

$$\text{(A22)}\qquad \begin{aligned} \Delta \varrho u \cdot \frac{dy}{ds} - \Delta \varrho v \cdot \frac{dx}{ds} &= 0, \\ \Delta u \cdot \frac{dx}{ds} + \Delta v \cdot \frac{dy}{ds} &= 0 \end{aligned}$$

where Δ indicates the jump across the shock and the derivatives are along the shock path. Equations (A22) may be written as

$$\text{(A23)}\qquad \Delta \frac{d\Psi}{ds} = \Delta \frac{d\Phi}{ds} = 0.$$

Since the unperturbed flow $\mathscr{F}_0$ is shockless and thus φ and ψ have continuous derivatives it follows that $\omega = \left.\frac{\partial \Phi}{\partial \tau}\right|_{\tau=0}$ and $\omega^* = \left.\frac{\partial \Psi}{\partial \tau}\right|_{\tau=0}$ satisfy

$$\text{(A24)}\qquad \Delta \frac{d\omega}{ds} = \Delta \frac{d\omega^*}{ds} = 0.$$

In terms of θ and σ, these equations may be rewritten as

$$\text{(A25)}\qquad \begin{aligned} 0 &= \Delta \omega_\theta \frac{d\theta}{ds} + \Delta \omega_\sigma \frac{d\sigma}{ds}, \\ 0 &= \Delta \omega_\theta^* \frac{d\theta}{ds} + \Delta \omega_\sigma^* \frac{d\sigma}{ds} \end{aligned}$$

and using (23) we obtain

$$\text{(A26)}\qquad \begin{aligned} 0 &= \Delta \omega_\theta \frac{d\theta}{ds} + \Delta \omega_\sigma \frac{d\sigma}{ds}, \\ 0 &= \Delta \omega_\sigma \frac{d\theta}{ds} - K \Delta \omega_\theta \frac{d\sigma}{ds}. \end{aligned}$$

Since $\Delta\omega_\theta$ and $\Delta\omega_\sigma$ are not both zero it follows that

$$\text{(A27)}\qquad d\theta^2 + K d\sigma^2 = 0,$$

that is, the shock path is characteristic and furthermore the derivative of ω along the characteristic is continuous.

Suppose there exist several weak shock in the flows $\mathscr{F}$ and that $\mathscr{F}_0$ has

no shocks, see §1. Then ω will be a solution of Problem III except that either (14) holds or (A26) and (A27).

Let Ω_σ be the domain in the θ, σ-plane described in §3 but slit along the shock paths. Consider the integral

$$\iint_{\Omega_\delta} (B\omega_\theta + C\omega_\sigma)(K\omega_{\theta\theta} + \omega_{\sigma\sigma})\, d\theta d\sigma$$

and integrate by parts. Equations (38) and (39) will be changed only by the addition of the line integrals I_s along both sides of the shock. By the characteristic condition (A27) this integral may be written as

$$\begin{aligned} I_s &= \oint (-B\, d\sigma - C\, d\theta)(\omega_\sigma \pm \sqrt{-K}\omega_\theta)^2 \\ &= \oint (-B\, d\sigma - Cd\theta) \left(\frac{d\omega}{ds}\right)^2 \Big/ \left(\frac{d\sigma}{ds}\right)^2, \end{aligned}$$

where the integral is taken around the slit. Since by (A24), $d\omega/ds$ is continuous across the shock it follows that $I_s = 0$. Therefore equations (38) and (39) hold and again $\omega_0 \equiv \omega_\sigma \equiv 0$ or $\omega^* = 0$ in Ω. The same argument also yields the same result for the domain Ω^*. This means that the perturbation problem is still incorrectly posed even if weak shocks are admitted.

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Received May, 1955.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. IX, 69–84 (1956)

On the Motion of Small Particles in a Potential Field of Flow

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1. Introduction

The analysis of the motion of small particles which are suspended in a moving fluid is of importance for the problems of aircraft icing, and it may also assist in the understanding of the silting up of rivers and estuaries. There exists a considerable body of work on this subject, both published and unpublished (see the list of references at the end of the paper). There are individual differences as to the equations by which the motion of the particles is supposed to be governed, but all the papers on this problem which are known to the writer involve the computation, for the most part step-by-step, of the trajectories of the individual particles.

In the present paper, we assume that the equation of motion of a particle is given by Stokes' law, or more generally, that the fluid exerts on the particle a drag force which is proportional to the vector difference between the velocity of the particle and the fluid velocity (compare [1, 2]). However, instead of confining ourselves to the trajectory of an individual particle, we consider the (virtual) field of flow which would be produced by a continuous distribution of small particles of given size each governed by the specified equation of motion. We derive various general properties of this virtual field of flow, including a counterpart of Kelvin's theorem on circulation. Moreover, we obtain a compact formula for the total mass of the particles deposited on a slender obstacle or on a low elevation in a plane wall.

2. General Analysis

Let

$$\phi_0 = \phi_0(x, y, z, t)$$

be the velocity potential of the field of flow F_0 of a compressible or incompressible fluid, where $x, y, z,$ denote the space coordinates and t denotes the time. Let

$$\mathbf{q}_0(x, y, z, t) = (u_0, v_0, w_0) = -\operatorname{grad} \phi_0$$

be the corresponding velocity vector.

Let m be the mass and $\mathbf{r} = (x, y, z)$ the position vector of a small particle which moves in the given field of flow, F_0. We assume that, owing to its smallness, the particle does not interfere with the field of flow of the fluid, and that its equation of motion is given by

$$m\frac{d^2\mathbf{r}}{dt^2} = -k(\mathbf{q} - \mathbf{q}_0) \tag{2.1}$$

where

$$\mathbf{q} = (u, v, w) = \frac{d\mathbf{r}}{dt}$$

is the velocity of the particle and k is a constant. In particular, if the particle has the shape of a sphere of radius R, then according to Stokes' law

$$k = 6\pi\mu R \tag{2.2}$$

where μ is the coefficient of viscosity of the fluid.

(2.1) represents a system of three ordinary differential equations of the second order for the quantities x, y, z regarded as functions of t. Thus, the solution of (2.1) is made determinate by the specification of the six quantities

$$x, y, z, u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt} \tag{2.3}$$

for a given time $t = t_1$.

Now suppose that the functions

$$u_1 = u_1(x, y, z), \quad v_1 = v_1(x, y, z) \quad w_1 = w_1(x, y, z) \tag{2.4}$$

are specified in some region of space. We may then regard x, y, z, u_1, v_1, w_1 as a set of initial values (2.3) for arbitrary but fixed $t = t_1$, so that (2.1) determines x, y, z, u, v, w for subsequent times t. Hence $\mathbf{q} = (u, v, w)$ may be regarded as the velocity vector of a (virtual) fluid, where in particular $\mathbf{q} = \mathbf{q}_1 = (u_1, v_1, w_1)$ for $t = t_1$.

Also, in this sense, we may replace (2.1) by

$$m\frac{D\mathbf{q}}{Dt} = -k(\mathbf{q} - \mathbf{q}_0) \tag{2.5}$$

where $D\mathbf{q}/Dt$ denotes differentiation following the motion of the virtual fluid. In scalar notation

$$m\frac{Du}{Dt} = m\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -k(u - u_0), \text{ etc.} \tag{2.6}$$

We shall denote the field of flow of the virtual fluid by F.

Now let C be a closed curve in the field F, and let Γ and Γ_0 be the circulation round C of the velocity vectors $\mathbf{q}$ and $\mathbf{q}_0$,

$$\Gamma = \int_C \mathbf{q}d\mathbf{s} = \int_C udx + vdy + wdz,$$

$$\Gamma_0 = \int_C \mathbf{q}_0 d\mathbf{s} = \int_C u_0dx + v_0dy + w_0dz.$$

We consider the rate of change of Γ in time as C moves with the field F (i.e., with the virtual fluid). Then

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\int_C \mathbf{q}d\mathbf{s} = \int_C \frac{D\mathbf{q}}{Dt}\,d\mathbf{s} + \int_C \mathbf{q}d\left(\frac{D\mathbf{s}}{Dt}\right).$$

Now, by (2.5)

$$\int_C \frac{D\mathbf{q}}{Dt}\,d\mathbf{s} = -\frac{k}{m}\int_C (\mathbf{q}-\mathbf{q}_0)d\mathbf{s} = -\frac{k}{m}\,(\Gamma - \Gamma_0)$$

while

$$\int_C \mathbf{q}d\,\frac{D\mathbf{s}}{Dt} = \int_C \mathbf{q}d\mathbf{q} = \tfrac{1}{2}\int_C d(q^2) = 0.$$

Hence

$$\frac{d\Gamma}{dt} = -\frac{k}{m}\,(\Gamma - \Gamma_0). \tag{2.7}$$

Moreover, we shall assume that Γ_0 is a constant, and hence we can replace (2.7) by

$$\frac{d}{dt}\,(\Gamma - \Gamma_0) = -\frac{k}{m}\,(\Gamma - \Gamma_0),$$

$$\Gamma = \Gamma_0 + \gamma e^{-kt/m} \tag{2.8}$$

where γ is a constant which may depend on C. By the theorems of Stokes and Kelvin, (2.8) holds provided only the field F_0 is irrotational in the region swept by C during the time under consideration. More particularly, if the field F_0 possesses a one-valued velocity potential ϕ_0, then $\Gamma_0 = 0$ and therefore

$$\Gamma = \gamma e^{-kt/m}. \tag{2.9}$$

It follows that *if the circulation Γ round the given curve C equals zero at any time then $\Gamma = 0$ always.*

If we are given initially only the conditions (2.3) for a single particle at time $t = t_1$, then we can always assume $\Gamma = 0$ at time $t = t_1$, and hence

at all subsequent times by defining that the velocities be equal to the constants u, v, w specified by (2.3) everywhere in the field F at time $t = t_1$. More generally, we have shown that if the virtual field of flow F is irrotational at any one time, then it is always irrotational, and hence, possesses a velocity potential ϕ. It will be assumed from now on, that this condition is indeed satisfied. Thus

$$\text{curl } \mathbf{q} = 0, \qquad \mathbf{q} = -\text{grad } \phi. \tag{2.10}$$

It follows that we may replace (2.6) by

$$m\left(-\frac{\partial^2 \phi}{\partial x \partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right) = -k(u - u_0) \tag{2.11}$$

with two similar equations. Integrating, we obtain the following counterpart of Bernoulli's equation:

$$m\left\{-\frac{\partial \phi}{\partial t} + \tfrac{1}{2}\left(\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right)\right\} = k(\phi - \phi_0) + H(t). \tag{2.12}$$

Conversely, it is not difficult to verify that if ϕ is a solution of (2.12), for specified ϕ_0, then $\mathbf{q} = -\text{grad } \phi$ is a solution of (2.5).

For steady conditions, we obtain

$$\tfrac{1}{2}m\left(\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right) = k(\phi - \phi_0) + H \tag{2.13}$$

where H may now be taken to be independent of the time.

3. Flow Past an Obstacle

Equation (2.13) may also be written as

$$\tfrac{1}{2}m(u^2 + v^2 + w^2) = k(\phi - \phi_0) + H. \tag{3.1}$$

Put

$$\phi' = \phi - \phi_0, \qquad u' = u - u_0 = -\frac{\partial \phi'}{\partial x}, \quad \text{etc.,}$$

then (3.1) yields

$$m(\tfrac{1}{2}q_0^2 + \tfrac{1}{2}q'^2 + u_0 u' + v_0 v' + w_0 w') = k\phi' + H \tag{3.2}$$

where

$$q_0 = \sqrt{u_0^2 + v_0^2 + w_0^2} \qquad q' = \sqrt{u'^2 + v'^2 + w'^2}.$$

We shall now suppose that q' is small compared with q_0. This amounts to the assumption that the difference between the velocity of the particle and the velocity of the fluid is of the nature of a (first order) correction to the

fluid velocity. For example, this assumption will be satisfied, generally speaking, in the case of uniform fluid flow past a slender obstacle, provided the particle is at rest relative to the fluid initially. The assumption breaks down in the close neighborhood of a stagnation point but, as in thin aerofoil theory, the overall results may still be expected to be valid.

Omitting q'^2 in (3.2) we obtain for the steady case,

$$m(u_0u' + v_0v' + w_0w') = k\phi' + H - \tfrac{1}{2}q_0^2. \tag{3.3}$$

Now the equations of a streamline in the field F_0 are

$$\frac{dx}{ds} = \frac{u_0}{q_0}, \quad \frac{dy}{ds} = \frac{v_0}{q_0}, \quad \frac{dz}{ds} = \frac{w_0}{q_0}$$

where s measures the arc of the streamline. Also, differentiating ϕ_0 with respect to s, we obtain

$$\frac{d\phi_0}{ds} = -q_0.$$

Hence, along the given streamline

$$\frac{dx}{d\phi_0} = -\frac{u_0}{q_0^2}, \quad \frac{dy}{d\phi_0} = -\frac{v_0}{q_0^2}, \quad \frac{dz}{d\phi_0} = -\frac{w_0}{q_0^2}.$$

Accordingly, we may modify the left hand side of (3.3) in the following way:

$$m(u_0u' + v_0v' + w_0w') = mq_0^2\left(\frac{dx}{d\phi_0}\frac{\partial\phi'}{\partial x} + \frac{dy}{d\phi_0}\frac{\partial\phi'}{\partial y} + \frac{dz}{d\phi_0}\frac{\partial\phi'}{\partial z}\right) = mq_0^2\frac{d\phi'}{d\phi_0}.$$

Inserting this in (3.3), we obtain

$$mq_0^2\frac{d\phi'}{d\phi_0} = k\phi' + H - \tfrac{1}{2}mq_0^2 \tag{3.4}$$

which is an ordinary linear differential equation of the first order for ϕ' as a function of ϕ along any given streamline. The general integral of (3.4) is

$$\phi' = \exp\left\{\int\frac{kd\phi_0}{mq_0^2}\right\}\int\left[\left(\frac{H}{mq_0^2} - \frac{1}{2}\right)\exp\left\{-\int\frac{kd\phi_0}{mq_0^2}\right\}\right]d\phi_0. \tag{3.5}$$

Now for the case of uniform fluid flow past a slender obstacle, with main stream velocity U in the direction of the x-axis, we may write

$$\phi_0 = -Ux + \phi_0', \quad u_0 = U - \frac{\partial\phi_0'}{\partial x},$$

$$v_0 = -\frac{\partial\phi_0'}{\partial y}, \quad w_0 = -\frac{\partial\phi_0'}{\partial z}.$$

The induced velocities $-\frac{\partial \phi_0'}{\partial x}, -\frac{\partial \phi_0'}{\partial y}, -\frac{\partial \phi_0'}{\partial z}$ are small compared with U (except near a stagnation point) and therefore

$$q_0^2 = \left(U - \frac{\partial \phi_0'}{\partial x}\right)^2 + \left(\frac{\partial \phi_0'}{\partial y}\right)^2 + \left(\frac{\partial \phi_0'}{\partial z}\right)^2 \doteqdot U^2 - 2U\frac{\partial \phi_0'}{\partial x} \tag{3.6}$$

where we have omitted terms of the second order of smallness compared with U^2. Also, if the particle is at rest relative to the fluid far upstream of the obstacle, then

$$u = U, \qquad v = w = 0, \qquad \phi' = \phi - \phi_0 = 0$$

in that region, and so, applying (3.1),

$$\tfrac{1}{2}mU^2 = H.$$

Hence

$$\frac{H}{mq_0^2} - \tfrac{1}{2} \doteqdot \tfrac{1}{2}\left(\frac{U^2}{U^2 - 2U\dfrac{\partial \phi_0'}{\partial x}} - 1\right) \doteqdot \frac{1}{U}\frac{\partial \phi_0'}{\partial x}.$$

Accordingly, (3.5) becomes approximately

$$\phi' = \exp\left\{\int \frac{kd\phi_0}{mU^2}\right\} \int \left[\frac{1}{U}\frac{\partial \phi_0'}{\partial x} \exp\left\{-\int \frac{kd\phi_0}{mU^2}\right\}\right] d\phi_0 .$$

The constant in the integral $\int (k/mU^2)\, d\phi_0$ is at our disposal, and so we may choose

$$\int \frac{kd\phi_0}{mU^2} = \lambda \phi_0$$

where $\lambda = k/mU^2$. Then

$$\phi' = \frac{1}{U} e^{\lambda \phi_0} \int \left(U + \frac{\partial \phi_0}{\partial x}\right) e^{-\lambda \phi_0}\, d\phi_0 \tag{3.7}$$

since

$$\frac{\partial \phi_0'}{\partial x} = U + \frac{\partial \phi_0}{\partial x}.$$

To insert the appropriate limits of integration in (3.7), we bear in mind that as x tends to $-\infty$, $\phi_0 \to \infty$ and $\phi' \to 0$. Hence

$$\phi' = \frac{1}{U}\int_{\infty}^{\phi_0} \left(U + \frac{\partial \Phi_0}{\partial x}\right) e^{-\lambda(\Phi_0 - \phi_0)}\, d\Phi_0 , \tag{3.8}$$

where we have used Φ_0 to distinguish the variable of integration, or

$$\phi = \phi_0 - \frac{1}{U}\int_{\phi_0}^{\infty}\left(U + \frac{\partial \Phi_0}{\partial x}\right)e^{-\lambda(\Phi_0-\phi_0)}\, d\Phi_0 . \tag{3.9}$$

Once again, we may simplify the analysis by taking into account that the streamlines of F_0 are nearly parallel to the x-axis and that, in the first approximation,

$$\phi_0 = -\,Ux, \qquad \frac{\partial \phi_0}{\partial x} = -\,U.$$

Hence

$$\phi = \phi_0 - \int_{-\infty}^{x}\left(U + \frac{\partial \Phi_0}{\partial X}\right) e^{-U\lambda(x-X)}\, dX, \tag{3.10}$$

where we use X to indicate the variable of integration. However, this additional simplification will not be required for the analysis of the next section.

4. Incompressible Flow in Two Dimensions

Let us now assume that the fluid flow is two-dimensional and incompressible, and takes place in planes parallel to the x, y-plane. We may then use z to denote the complex variable $x + iy$. Also, ϕ_0 now is the real part of a complex potential w_0,

$$w_0 = \phi_0 + i\psi_0$$

and we may write

$$W_0 = \Phi_0 + i\Psi_0$$

where Φ_0 is the variable of integration in (3.9). But the integration in (3.9) is performed along a streamline, and so $\Psi_0 = \text{const} = \psi_0$, throughout. Also

$$\frac{\partial \phi_0}{\partial x} = \mathcal{R}e\left(\frac{dw_0}{dz}\right)$$

where dw_0/dz is an analytic function of z, and hence of w_0. Thus

$$\int_{\phi_0}^{\infty}\left(U + \frac{\partial \Phi_0}{\partial x}\right) e^{-\lambda(\Phi_0-\phi_0)}\, d\Phi_0 = \mathcal{R}e \int_{w_0}^{\infty}\left(U + \frac{dW_0}{dz}\right) e^{-\lambda(W_0-w_0)}\, dW_0$$

and therefore

$$\phi = \mathcal{R}e\left\{w_0 - \frac{1}{U}\int_{w_0}^{\infty}\left(U + \frac{dW_0}{dz}\right) e^{-\lambda(W_0-w_0)}\, dW_0\right\}. \tag{4.1}$$

If the upper limit of the integral in (4.1) were a fixed finite complex number, it would follow immediately that the expression within the curly brackets is an analytic function of w_0, and some reflection shows that this is true even in the present case. Hence ϕ is the real part of a complex potential w, where

(4.2) $$w = w_0 - \frac{1}{U}\int_{w_0}^{\infty}\left(U + \frac{dW_0}{dz}\right) e^{-\lambda(W_0 - w_0)}\, dW_0$$

or

(4.3) $$w(z_0) = w_0(z_0) + \frac{1}{U}\int_{-\infty}^{z_0}\left(U + \frac{dW_0}{dz}\right)\frac{dW_0}{dz}\, e^{-\lambda(W_0 - w_0(z_0))}\, dz.$$

The lower limit of the integral in (4.3), $-\infty$, indicates that the integral is to be taken from a point far upstream of the obstacle. Apart from this, the complex representation has the advantage that the path of integration may now be deformed at will in any finite region of the field F_0.

5. The Mass Flow of Particles

Assume now that particles of equal size (i.e. of equal m and k) are distributed uniformly and densely in the region far upstream of the obstacle. We may then define the density ϱ of our virtual fluid as the average total mass of particles per unit volume. The variation of ϱ following the motion of the fluid is given, as usual, by the equation of continuity

(5.1) $$\frac{1}{\varrho}\frac{D\varrho}{Dt} = -\operatorname{div} \mathbf{q}.$$

But under the conditions of the preceding section, ϕ is the real part of a complex analytic function and, therefore, satisfies Laplace's equation

$$\Delta\varphi = -\operatorname{div} \mathbf{q} = 0.$$

Hence

(5.2) $$\frac{D\varrho}{Dt} = 0,$$

showing that the virtual fluid is incompressible.

Equation (5.2) is only approximately true. It can be proved more rigorously that if the real fluid is irrotational and incompressible, and if the virtual fluid is at rest initially (at time $t = 0$, say) then

(5.3) $$\operatorname{div} \mathbf{q} \leqq 0.$$

This shows that ϕ is a subharmonic function,

(5.4) $$\Delta\phi = -\operatorname{div} \mathbf{q} \geqq 0.$$

In order to establish (5.3), we apply the operator div to (2.5). This yields, on the right hand side

(5.5) $$-k \operatorname{div} (\mathbf{q} - \mathbf{q}_0) = -k \operatorname{div} \mathbf{q},$$

while on the left hand side

$$m \operatorname{div}\left(\frac{D\mathbf{q}}{Dt}\right) = m\left(\frac{D}{Dt}\operatorname{div}\mathbf{q} + \Delta\right) \tag{5.6}$$

where

$$\Delta = \left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y} + \left(\frac{\partial w}{\partial z}\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2.$$

Thus, Δ can be represented by a sum of squares and therefore

$$\Delta \geqq 0.$$

In deriving this relation, we have made use of the fact that curl $\mathbf{q} = 0$, by (2.9). Combining (5.5) and (5.6), we obtain

$$\frac{D}{Dt}\operatorname{div}\mathbf{q} + \frac{k}{m}\operatorname{div}\mathbf{q} = -m\Delta. \tag{5.7}$$

This is an ordinary differential equation for div $\mathbf{q}$, following the motion of the virtual fluid, and since the virtual fluid is supposed to be at rest for $t = 0$, we have the initial condition

$$\operatorname{div}\mathbf{q} = 0 \qquad \text{for } t = 0.$$

With this condition, the solution of (5.7) is

$$\operatorname{div}\mathbf{q} = -me^{-kt/m}\int_0^t \Delta e^{kt/m}\,dt. \tag{5.8}$$

Thus,

$$\operatorname{div}\mathbf{q} \leqq 0$$

as asserted by (5.3). It then follows from (5.1) that

$$\frac{D\varrho}{Dt} \geqq 0 \tag{5.9}$$

which shows that the density of the virtual fluid is a non-decreasing function of the time, following the motion of the fluid. However, in the sequel we shall adopt the approximate conclusion (5.2) according to which ϱ is actually constant. Then the mass flow of particles across a curve connecting two points A and B, in unit time, is given by the product of the density ϱ and of the difference $\psi_A - \psi_B$, where $\psi = \mathcal{I}m(w)$ is the stream function of the field F,

$$M = -\varrho\,\mathcal{I}m(w_B - w_A). \tag{5.10}$$

However, this formula becomes inapplicable if any of the streamlines of F which cross the curve connecting A and B have already crossed the solid boundary of the obstacle further upstream. Excluding this case, we assume more particularly, that A and B are situated on the same streamline of the field F_0. Then

$$\mathscr{I}m\,(w_0(z_B) - w_0(z_A)) = 0$$

where z_A, z_B are the complex numbers corresponding to A, B. Inserting (4.3) in (5.10), we have

$$M = -\frac{\varrho}{U}\mathscr{I}m\left\{e^{\lambda w_0(z_B)}\int_{-\infty}^{z_B}\left(U+\frac{dW_0}{dz}\right)\frac{dW_0}{dz}e^{-\lambda W_0}\,dz \right. \tag{5.11}$$
$$\left. - e^{\lambda w_0(z_A)}\int_{-\infty}^{z_A}\left(U+\frac{dW_0}{dz}\right)\frac{dW_0}{dz}e^{-\lambda W_0}\,dz\right\}$$

and modifying (5.11) slightly

$$M = -\frac{\varrho}{U}\mathscr{I}m\left\{\left(e^{\lambda w_0(z_B)} - e^{\lambda w_0(z_A)}\right)\int_{-\infty}^{z_A}\left(U+\frac{dW_0}{dz}\right)\frac{dW_0}{dz}e^{-\lambda W_0}\,dz \right. \tag{5.12}$$
$$\left. + e^{\lambda w_0(z_B)}\int_{z_A}^{z_B}\left(U+\frac{dW_0}{dz}\right)\frac{dW_0}{dz}e^{-\lambda W_0}\,dz\right\}.$$

We now consider flow without circulation round a symmetric obstacle at zero incidence; we take A as the front stagnation point and B as a point on the top surface of the obstacle. We may choose A as origin of coordinates, then the negative x-axis is a streamline; we may assume $\mathscr{I}m(w) = \psi_0 = 0$ along that line. It follows that $\psi_0 = 0$ also on the top surface of the obstacle, more particularly between A and B. Also dW_0/dz is real on the negative x-axis and so the first term within the curly brackets of (5.12) is real. Accordingly, the expression for M reduces to

$$M = -\frac{\varrho}{U}\mathscr{I}m\left\{e^{\lambda w_0(z_B)}\int_{0}^{z_B}\left(U+\frac{dW_0}{dz}\right)\frac{dW_0}{dz}e^{-\lambda W_0}\,dz\right\} \tag{5.13}$$
$$= -\frac{\varrho}{U}e^{\lambda w_0(z_B)}\int_{0}^{z_B}\mathscr{I}m\left\{U+\frac{dW_0}{dz}\right\}\frac{dW_0}{dz}e^{-\lambda W_0}\,dz$$

since

$$\frac{dW_0}{dz}e^{-\lambda W_0}\,dz = e^{-\lambda W_0}\,dW_0$$

is real along the top surface of the obstacle. But

$$\mathscr{I}m\left\{U+\frac{dw_0}{dz}\right\} = v_0 = -\frac{\partial\phi_0}{\partial y}$$

and, since the obstacle is slender, we have the linearized boundary condition

$$v_0 = Uf'(x)$$

where $y = f(x)$ is the equation of the top surface. Hence (5.13) may be replaced by

$$M = -\varrho \int_0^{z_B} f'(x) e^{-\lambda(W_0 - w_0(z_B))} \frac{dW_0}{dz} dz. \tag{5.14}$$

Finally, replacing $W_0 - w_0(z_B)$ and $(dW_0/dz)dz$ by the first approximations $-U(x - x_B)$ and $-Udx$, and bearing in mind that $\lambda = k/mU^2$ we obtain

$$M = \varrho U \int_0^{x_B} f'(x) e^{-k(x_B - x)/mU} dx \tag{5.15}$$

where $x_B = \mathcal{R}e\,(z_B)$. This is a compact formula for the total mass of the particles which strike the top surface of the obstacle between A and B, in unit time. This formula applies equally to an unstaggered cascade of geometrically similar symmetric obstacles at zero incidence, and to an elevation in a straight wall (e.g. river bottom) which is parallel to the main stream.

Using (3.10) we might have arrived at the final formula (5.15) also by a less careful analysis.

6. Examples

Write

$$\xi = x/x_B\,, \qquad \tau = f(x_B)/x_B\,, \qquad g(\xi) = f(x_B\xi)/f(x_B).$$

Then

$$f'(x) = f'(\xi)f(x_B)/x_B = \tau g'(\xi).$$

Substituting in (5.15), we obtain

$$\begin{aligned} M &= \varrho U \tau x_B \int_0^1 g'(\xi) e^{-\sigma(1-\xi)}\, d\xi \\ &= \varrho U f(x_B) \int_0^1 g'(\xi) e^{-\sigma(1-\xi)}\, d\xi \end{aligned} \tag{6.1}$$

where σ is the non-dimensional parameter kx_B/mU. As σ tends to zero, M tends to

$$M_0 = \varrho U f(x_B) \int_0^1 g'(\xi)\, d\xi = \varrho U f(x_B). \tag{6.2}$$

This is in agreement with the fact that for vanishingly small viscosity the particles retain the main stream velocity, so that M becomes equal to the product of density, main stream velocity, and frontal area. Accordingly, it is appropriate to restate the general result in terms of the coefficient

$$\eta = \frac{M}{M_0} = \int_0^1 g'(\xi) e^{-\sigma(1-\xi)}\, d\xi. \tag{6.3}$$

Suppose first that the front portion of the obstacle has the shape of a wedge,

$$f(x) = cx, \qquad c > 0.$$

The shape of the rear portion of the obstacle is irrelevant, but we may assume, for example, that we are dealing with a double-wedge aerofoil. Then

$$f(x_B) = cx_B, \qquad g(\xi) = cx\xi_B/cx_B = \xi$$

and hence

$$\eta = \int_0^1 e^{-\sigma(1-\xi)}\, d\xi = e^{-\sigma}\left[\frac{1}{\sigma} e^{\sigma\xi}\right]_0^1 = \frac{1}{\sigma}(1 - e^{-\sigma}). \tag{6.4}$$

Next, consider the case of an obstacle of elliptic shape which is given by the equation

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{6.5}$$

Then

$$y = f(x) = b\sqrt{1 - \frac{(x-a)^2}{a^2}}$$

and the front stagnation point is located at the origin, as required. We choose x_B at the midchord of the obstacle, $x_B = a$, so that $f(x_B) = b$. Then

$$g(\xi) = b\sqrt{1 - \frac{(a\xi - a)^2}{a^2}} \Big/ b = \sqrt{1-(1-\xi)^2},$$

$$g'(\xi) = \frac{1-\xi}{\sqrt{1-(1-\xi)^2}},$$

$$\eta = \int_0^1 \frac{1-\xi}{\sqrt{1-(1-\xi)^2}} e^{-\sigma(1-\xi)}\, d\xi = \int_0^1 \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\sigma\zeta}\, d\zeta. \tag{6.6}$$

It is not difficult to verify that along the entire segment AB of the contour of the obstacle, the streamlines of F cross into the obstacle from upstream so that we are dealing with a real case. The integral in (6.6) can be evaluated in terms of standard transcendental functions (see [6] p. 136, No. 28)

$$\eta = 1 - \frac{\pi}{2}\left(I_1(\sigma) - \mathbf{L}_1(\sigma)\right). \tag{6.7}$$

In this formula, $I_1(\sigma)$ is a modified Bessel function of the first kind,

$$I_1(\sigma) = \sum_{n=0}^{\infty} \frac{(\sigma/2)^{2n+1}}{\Gamma(n+1)\Gamma(n+2)},$$

and $\mathbf{L}_1(\sigma)$ is a modified Struve function,

$$\mathbf{L}_1(\sigma) = \sum_{n=0}^{\infty} \frac{(\sigma/2)^{2n+2}}{\Gamma(n+\frac{3}{2})\Gamma(n+\frac{5}{2})}.$$

Table 1 shows values of η for various values of σ, for wedge and ellipse. For a spherical particle, k is given by (2.2), according to Stokes' law, while $m = (4/3)\pi\varrho' R^3$, where ϱ' is the density of the particle. Hence

$$\sigma = \frac{k}{m}\frac{x_B}{U} = \frac{9\mu x_B}{2\varrho' R^2 U}.$$

If the particle is a water droplet, and the surrounding medium is air, then we may take $\varrho' = 1$, $\mu = 1.7 \times 10^{-4}$ in c.g.s. units. Typical values for x_B, R, U are, in the same units,

$$x_B = 3 \times 10^2, \qquad R = 3 \times 10^{-3}, \qquad U = 1.5 \times 10^4$$

for the aircraft icing problem. This yields $\sigma = 1.7$.

TABLE 1

	η (wedge)	η (ellipse)
0.0	1.000	1.000
0.2	0.906	0.855
0.4	0.824	0.734
0.6	0.752	0.630
0.8	0.688	0.543
1.0	0.632	0.468
2.0	0.432	0.310

Although no direct comparison is possible, our results appear to be consistent with [2]. The fact that our approximations break down in the neighborhood of the front stagnation point does not invalidate these results. To see this, we shift the point A a short way upstream, and remove the path of integration AB from the neighborhood of the front stagnation point. Then our approximations may be assumed to hold uniformly on, and upstream of, the modifed AB. However, the actual flow across AB is the same as before, by continuity, and the mathematical result, given e.g. by (5.13), has not changed, by Cauchy's theorem. This argument shows that our overall results may be expected to hold in spite of the breakdown of our assumptions near the front stagnation point.

7. The Influence of Gravity

The influence of gravity has been excluded from our considerations so far. Suppose now that a conservative body force, which is given by the potential Ω, acts on the particle. Then (2.1) and (2.5) have to be replaced by

$$m\frac{d^2\mathbf{r}}{dt^2} = -k(\mathbf{q}-\mathbf{q}_0) - m\,\mathrm{grad}\,\Omega \tag{7.1}$$

and

$$m\frac{D\mathbf{q}}{Dt} = -k(\mathbf{q}-\mathbf{q}_0) - m\,\mathrm{grad}\,\Omega, \tag{7.2}$$

respectively. Now

$$\int_C \mathrm{grad}\,\Omega\,d\mathbf{s} = 0$$

when the integral is taken round a closed curve C, for one-valued Ω. Hence (2.8) and its general consequences still hold. In place of (2.12) and (2.13) we obtain

$$m\left\{-\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right)\right\} = k(\phi-\phi_0)-m\Omega+H(t) \tag{7.3}$$

and

$$\frac{1}{2}m\left(\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right) = k(\phi-\phi_0)-m\Omega+H. \tag{7.4}$$

Moreover, if Ω is a harmonic potential,

$$\Delta\Omega = \mathrm{div}\,\mathrm{grad}\,\Omega = 0,$$

then, by applying the operation div to (7.2), we may still deduce (5.3) under the conditions stated in Section 5.

Let us now consider the particular case of flow past an obstacle under two-dimensional conditions, as in Section 4. We suppose that the conservative body force is the force of gravity acting in a direction perpendicular to the free stream. Then

$$\Omega = gy \tag{7.5}$$

provided we can neglect the buoyancy of the particle. If we wish to take the buoyancy into account, we only have to replace g in (7.5) by

$$g' = \left(1-\frac{m'}{m}\right)g$$

where m is the mass of the particle, as before, and m' is the mass of the fluid displaced by it. We then obtain from (7.4)

$$\frac{1}{2}m\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]=k(\phi-\phi_0)-mg'y+H \tag{7.6}$$

for steady conditions. If no obstacle is present, $\phi_0 = -Ux$, and (7.6) is satisfied by

$$\phi = -Ux + \frac{m}{k}g'y \tag{7.7}$$

which corresponds to the steady fall of the particles with „terminal velocity"

$$V = -\frac{m}{k}g' = -\frac{m-m'}{k}g. \tag{7.8}$$

The corresponding value of H is

$$H = \tfrac{1}{2}m(U^2+V^2) \tag{7.9}$$

and this must also be the value of H for steady flow past an obstacle, as can be seen by considering conditions far upstream. For that case, we write

$$\phi = -Ux + \frac{m}{k}g'y + \phi' = -Ux + Vy + \phi'$$

so that

$$u = U - \frac{\partial\phi'}{\partial x}, \qquad v = -V - \frac{\partial\phi'}{\partial y}.$$

Then (7.6) becomes

$$\frac{1}{2}m\left[\left(\frac{\partial\phi'}{\partial x}\right)^2+\left(\frac{\partial\phi'}{\partial y}\right)^2-2U\frac{\partial\phi'}{\partial x}+2V\frac{\partial\phi'}{\partial y}\right]=k(\phi'-\phi_0') \tag{7.10}$$

where ϕ_0' is the induced velocity potential of the fluid flow, $\phi_0' = \phi_0 + Ux$.

We now assume that the terminal velocity V is small compared with the free stream velocity U and we may suppose that the same applies to the induced velocities in the field F, $-\frac{\partial\phi'}{\partial x}$ and $-\frac{\partial\phi'}{\partial y}$.

Then (7.10) becomes

$$mU\frac{\partial\phi'}{\partial x} + k\phi' = k\phi_0'. \tag{7.11}$$

The solution of this differential equation is, with the appropriate limits of integration,

$$\phi' = U\lambda e^{-U\lambda x}\int_{-\infty}^{x}\Phi_0' e^{U\lambda X}\,dX. \tag{7.12}$$

In this formula, $\lambda = k/mU^2$, as before, and we have used Φ_0' in order to indicate that the function depends on the variable of integration X. The

lower limit of integration has been chosen so as to ensure that ϕ' vanishes far upstream, as required.

Integrating by parts in (7.12),

$$\phi' = e^{-U\lambda x}\left\{\left[\Phi_0' e^{U\lambda X}\right]_{-\infty}^{x} - \int_{-\infty}^{x} \frac{\partial \Phi_0'}{\partial X} e^{U\lambda X}\, dX\right\}$$

$$= \phi_0' - \int_{-\infty}^{x} \frac{\partial \Phi_0'}{\partial X} e^{-U\lambda(x-X)}\, dX.$$

But

$$\phi = -Ux + Vy + \phi', \qquad \phi_0 = -Ux + \phi_0', \qquad \frac{\partial \phi_0'}{\partial x} = U + \frac{\partial \phi_0}{\partial x}$$

and thus

$$\phi = \phi_0 - \int_{-\infty}^{x}\left(U + \frac{\partial \Phi_0}{\partial x}\right) e^{-U\lambda(x-X)}\, dX + Vy. \tag{7.13}$$

Comparing (7.13) with (3.1), we see that, within the stated approximations, the velocity field due to gravity is additive to the field calculated previously. The result still represents the field of flow of an incompressible fluid. Thus, in order to obtain the total mass of the particles which strike a segment AB of the contour of the obstacle, we only have to add to (5.15) the mass which would strike the obstacle in unit time if it were at rest. For the examples calculated in Section 6, this quantity is given by $\varrho V x_B$.

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Received February, 1955.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. IX, 85–91 (1956)

On a Generalization of the Normal Basis in Abelian Algebraic Number Fields*

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It is known [1, 2] that in certain normal algebraic number fields a so-called „normal" basis can be chosen for the integers which consists of the conjugates of one number. Let F be the field in question, and assume that it is of degree n and that $\alpha^{(S_i)}$, $i = 1, 2, \cdots, n$, is a normal basis for F, where the S_i are the elements of the Galois group of F.

With any integral basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ one considers the matrix $D = (\alpha_i^{(S_k)})$, $i, k = 1, 2, \cdots, n$. Since the square of the determinant of D is the discriminant of F we call D a discriminant matrix. Consider this matrix for a normal basis. We then have

$$D = (\alpha^{(S_i S_k)}), \qquad i, k = 1, 2, \cdots, n.$$

This special D is symmetric if and only if the field is abelian. For complex matrices symmetry is here used in the ordinary sense, not in the hermitian.

It will be shown in §1 that for an abelian field this special D is always normal.[1]

The main question (which is considered in §3) is whether there is any other possibility for the discriminant matrix to be normal.[2] It is shown that for cyclic fields there is no other possibility.

In §2 the uniqueness of the normal basis is studied. It is shown that for $n = 2, 3, 4, 6$ the normal basis is unique apart from permutations, and that for all other n the normal basis is not unique.

In the general case the problem studied here leads to the study of group matrices; for, the discriminant matrix derived from a normal basis is a permutation of a group matrix. In the case of cyclic fields we are led to circulants.

§1. THEOREM 1. *Let F be an abelian field of degree n with a normal basis $\alpha^{(S_i)}$ and D the discriminant matrix corresponding to this basis. Then D is normal.*

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*This work was supported (in part) by the Office of Naval Research.

[1]A matrix A is called normal if $AA^* = A^*A$ where $A^* = \bar{A}'$, the transposed conjugate of A. A complex matrix A with $A = A'$ is in general not normal.

[2]See [3].

Proof: As mentioned in the introduction, it is clear that D is symmetric for abelian fields. It remains to show that D is normal even if F is a complex field. The (i, k) element of DD^* is

$$\sum_{r=1}^{n} \alpha^{(S_i S_r)} \overline{\alpha}^{(S_k S_r)}.$$

The corresponding element of D^*D is the conjugate complex value. It will now be shown that the element is real. Let $\alpha^{(S_i S_r)} \overline{\alpha}^{(S_k S_r)}$ be complex for some value of r. We will see that the sum also contains its conjugate. Among the S_i there is a substitution S_ω with $S_\omega^2 = 1$ such that $\varrho^{(S_\omega)} = \bar{\varrho}$ for all ϱ in F. We then have by virtue of the commutativity of the Galois group of F

$$\begin{aligned} \overline{\alpha}^{(S_i S_r)} \alpha^{(S_k S_r)} &= \overline{\alpha}^{(S_\omega^2 S_i S_r)} \alpha^{(S_\omega^2 S_k S_r)} \\ &= \alpha^{(S_i S_{r_1})} \overline{\alpha}^{(S_k S_{r_1})} \end{aligned}$$

where $S_{r_1} = S_\omega S_r$. This proves the theorem.

§2. Any integral basis is obtained from the given normal basis by means of a substitution (a_{ik}) where a_{ik} are rational integers and $|a_{ik}| = \pm 1$. When searching for an integral basis with normal discriminant matrix we may simply find a permutation of the given normal basis or another normal basis.

THEOREM 2. *Let F be a cyclic field of degree n with normal basis $\alpha^{(S_i)}$. If $n = 2, 3, 4, 6$, the normal basis is unique. For all other n the normal basis is not unique.*

Proof: Assume that $S_i = S^i$ where S is a generator of the cyclic Galois group of F. Consider a unimodular matrix (a_{ik}) of rational integers such that $(a_{ik})(\alpha^{(S^i)})$ is again a normal basis.

We prove first the following lemma:

LEMMA 1. *Let (a_{ik}) be a unimodular matrix of rational integers and $\alpha^{(S_i)}$ a normal basis in a cyclic field. In order for $(a_{ik})(\alpha^{(S_i)})$ to be a normal basis also it is necessary and sufficient that (a_{ik}) be the product of a circulant matrix and a permutation matrix.*

Proof: A circulant[3] matrix is usually assumed of the form

$$(a_1, a_2, \cdots, a_n) = \begin{bmatrix} a_1\, a_2 \cdots a_n \\ a_n\, a_1 \cdots a_{n-1} \\ \cdots\cdots\cdots \\ a_2\, a_3 \cdots a_1 \end{bmatrix}.$$

In the cyclic case the discriminant matrix of a normal basis multiplied on the left by a suitable permutation matrix becomes a circulant. Converse-

[3]For a detailed study of circulants see e.g. [4].

ly, if the discriminant matrix is a permutation of a circulant then the basis from which it is derived is normal.

Assume then that the discriminant matrix, multiplied by a suitable permutation matrix is the circulant $(\alpha^{(S^{i-k})})$. Use now the fact that the product of circulants and the inverse of a circulant is again a circulant. It follows that $(a_{ik})(\alpha^{(S^i)})$ will be again a normal basis if and only if (a_{ik}) is a permutation of a circulant.

From Lemma 1 it follows that the non-uniqueness of the normal basis depends on the existence of a non-trivial unimodular circulant of rational integers. (A circulant is called trivial if it has only one non-zero entry in each row). Theorem 2 is therefore a consequence of

THEOREM 2′. *An $n\times n$ circulant of rational integers cannot be unimodular for $n = 2, 3, 4, 6$ unless it is trivial. For all other values of n there are non-trivial unimodular circulants.*[4]

Proof: A circulant $(a_1, a_2, \cdots, a_n)$ is unimodular if and only if $a_1 + a_2\zeta + \cdots + a_n\zeta^{n-1}$ is a unit in the field generated by the n-th roots of 1, for each of the n values of ζ such that $\zeta^n = 1$. Since for $n = 2, 3, 4, 6$ the field of the n-th roots of unity does not contain any units of infinite order it follows that for these degrees all a_i but one are zero.

Suppose now that $n = 2^\mu n_1$, where n_1 is odd and $\geqq 5$. Then there is an odd k with $(k, n_1) = 1$ such that $3 \leqq k \leqq n_1$. For this k, it is easily shown that $\varepsilon = (1 + \zeta^{2^\mu k})/(1 + \zeta^{2^\mu})$ is a unit in the field of the n-th roots of 1, for each of the n values of ζ such that $\zeta^n = 1$. Thus the circulant based on ε is unimodular and non-trivial. This leaves only the n's of the form 2^μ, $3 \cdot 2^\mu$ for consideration. In view of the fact that if $a_1 + a_2\zeta + \cdots + a_n\zeta^{n-1}$ is a unit in the field of the n-th roots of 1 for each of the n values of ζ snch that $\zeta^n = 1$ then $a_1 + a_2(\zeta^{1/k})^k + \cdots + a_n(\zeta^{1/k})^{k(n-1)}$ is a unit in the field of the nk-th roots of 1 for each of the nk values of $\zeta^{1/k}$ such that $(\zeta^{1/k})^{nk} = 1$, it is only necessary to exhibit a non-trivial unimodular circulant for $n = 8$ and $n = 12$. For $n = 8$, the circulant $(1, 1, 0, -1, -2, -1, 0, 1)$ is unimodular and non-trivial. It is derived from the unit $(1 + \sqrt{2})^2$. For $n = 12$, the circulant $(3, 2, 1, 0, -1, -2, -2, -2, -1, 0, 1, 2)$ is unimodular and non-trivial. It is derived from the unit $(2 + \sqrt{3})^2$.

§3. We now deal with the main question. Denote the symmetric discriminant matrix of the fixed normal basis of an abelian field by D and put $(a_{ik})D = \Delta$. The problem is to find a unimodular matrix (a_{ik}) of rational integers such that $\Delta\Delta^* = \Delta^*\Delta$. We prove

[4] We are indebted to Dr. S. Gorn for turning our attention to the problem of the uniqueness of the normal basis. Recently H. P. F. Swinnerton-Dyer found another proof of Theorem 2′.

THEOREM 3. *Let F be an abelian field of degree n with a normal basis. In order for the discriminant matrix $\Delta = (a_{ik})D$ to be normal it is necessary and sufficient that $(a_{ik})'(a_{ik})$ be a group matrix corresponding to the Galois group of F.*

Proof[5]: The equation $\Delta\Delta^* = \Delta^*\Delta$ leads to

$$(a_{ik})DD^*(a_{ik})' = D^*(a_{ik})'(a_{ik})D.$$

We next note that DD^* is a matrix consisting of rational numbers. For, let $\alpha^{(S_i)}$ be the normal basis from which D is derived. Then $DD^* = (\alpha^{(S_iS_k)})(\overline{\alpha}^{(S_iS_k)})'$, i.e. the (i, k)-element of DD^* is $\sum_{r=1}^{n} \alpha^{(S_iS_r)}\overline{\alpha}^{(S_kS_r)}$. If we now replace α by $\alpha^{(S)}$ where S is any one of the S_r we obtain in virtue of the commutativity of the S_r

$$\sum_{r=1}^{n} \alpha^{(S_iSS_r)}\,\overline{\alpha}^{(S_kSS_r)}$$

which coincides with the previous expression since SS_r runs through all elements of the Galois group with S_r. Thus DD^* is rational.

Hence it follows that also the right side $D^*(a_{ik})'(a_{ik})D$ is unaltered if α in D is replaced by $\alpha^{(S)}$ where S is any of the elements of the Galois group of F. However, replacing α in D by $\alpha^{(S)}$ effects a permutation of the rows of D, i.e. can be written in the form P_SD, where P_S is a permutation matrix. We thus obtain

$$(*) \qquad P'_S(a_{ik})'(a_{ik})\,P_S = (a_{ik})'(a_{ik})$$

where the P_S are $n \times n$ permutation matrices which form a representation of the Galois group of F. In the case that F is cyclic the P_S can be taken as the powers of

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

It is then known that the previous relation implies that $(a_{ik})'(a_{ik})$ is a circulant. In the general abelian case it can be concluded that $(a_{ik})'(a_{ik})$ is a group matrix with respect to the Galois group of F. For, let P_S be the permutation $\begin{pmatrix} 1 & 2 & \cdots & n \\ s_1 & s_2 & \cdots & s_n \end{pmatrix}$. Then $(*)$ implies that the matrix obtained from $(a_{ik})'(a_{ik})$ by permuting the rows according to P_S is equal to the matrix obtained from $(a_{ik})'(a_{ik})$ by permuting the columns according to P_S^{-1}. It

[5]K. Goldberg had previously found a different proof in the case of cyclic fields.

follows that the s_1-th row of $(a_{ik})'(a_{ik})$ is obtained by applying the permutation P_S^{-1} to the first row. Since the Galois group is transitive, every row of $(a_{ik})'(a_{ik})$ is obtained from its first row in this manner.

We will now discuss the case of cyclic fields only.

It was shown in §2 that for $n = 2, 3, 4, 6$ only trivial circulants exist, hence for these values of n the circulant $(a_{ik})'(a_{ik})$ being symmetric is the unit matrix. This implies that (a_{ik}) is an orthogonal matrix; thus each row contains only one element (which is ± 1) different from 0 in our case of rational integral elements. Such a matrix will be called a generalized permutation matrix. We now show

THEOREM 4. *Let F be a cyclic field with normal basis. There exists no basis with normal discriminant matrix unless it is a normal basis or a generalized permutation of a normal basis.*

Theorem 4 follows from

THEOREM 4′. *A unimodular circulant of the form AA' where A is a matrix of rational integers, is equal to CC' where C is again a unimodular circulant of rational integers.*

Proof: We find it convenient to prove the following lemma first:

LEMMA 2. *Let n be a positive integer, and suppose that M is a generalized n-cycle with $\det M = \det P$. Then M and P are similar elements of the group of generalized permutation matrices.*

Proof: We show first that the generalized n-cycle

$$K = \begin{bmatrix} 0 & \sigma_1 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \sigma_{n-1} \\ \sigma_n & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \sigma_i = \pm 1,$$

is similar to P, if $\sigma_1\sigma_2 \cdots \sigma_n = 1$ (this is the necessary and sufficient condition that $\det K = \det P$). Put $E = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ where $\varepsilon_i \pm 1$ and the exact values of ε_i are to be determined later. Then

$$E^{-1}PE = EPE = \begin{bmatrix} 0 & \varepsilon_1\varepsilon_2 & 0 \cdots & 0 \\ 0 & 0 & \varepsilon_2\varepsilon_3 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots & \varepsilon_{n-1}\varepsilon_n \\ \varepsilon_n\varepsilon_1 & 0 & 0 \cdots & 0 \end{bmatrix}.$$

We seek ε_i such that

$$\begin{aligned}
\varepsilon_1\varepsilon_2 &= \sigma_1 ,\\
\varepsilon_2\varepsilon_3 &= \sigma_2 ,\\
&\cdots\cdots\\
\varepsilon_{n-1}\varepsilon_n &= \sigma_{n-1} ,\\
\varepsilon_n\varepsilon_1 &= \sigma_n .
\end{aligned}$$

Since $\sigma_1\sigma_2 \cdots \sigma_n = 1$, these equations have the solution

$$\begin{aligned}
\varepsilon_1 &= \pm 1,\\
\varepsilon_2 &= \sigma_1\varepsilon_1 ,\\
\varepsilon_3 &= \sigma_1\sigma_2\varepsilon_1 ,\\
&\cdots\cdots\cdots\cdots\\
\varepsilon_n &= \sigma_1\sigma_2 \cdots \sigma_{n-1}\varepsilon_1 .
\end{aligned}$$

We conclude therefore that $E^{-1}PE = K$, so that K and P are similar elements of the group of generalized permutation matrices.

Suppose now that M is a generalized n-cycle with $\det M = \det P$. We can write $M = HM_0$, where H is a diagonal matrix of $\pm$ 1's and M_0 is an n-cycle in the usual sense. Hence there is a permutation matrix Q such that $M_0 = Q'PQ$, and so $M = HQ'PQ$. Since Q' is a permutation matrix and H is a diagonal matrix of $\pm$ 1's there is some other diagonal matrix H_0 of $\pm$ 1's such that $HQ' = Q'H_0$. Thus $M = Q'H_0PQ$, which implies $\det H_0 = 1$, since $\det M = \det P$. Hence $\det H_0P = \det P$, and so H_0P and P are similar elements of the group of generalized permutation matrices. The same is therefore true for M and P, and Lemma 2 is proved.

We return now to Theorem 4′. Let A be a unimodular matrix of rational integers of order n such that AA' is a circulant. Then AA' commutes with P, so that $(PA)(PA)' = AA'$. Thus the matrix $A^{-1}PA$ is orthogonal, and since A and P have rational integral elements and are unimodular, $A^{-1}PA$ is a generalized permutation matrix. Put $A^{-1}PA = M$, so that $PA = AM$. We remark first that M is of period n, since M is similar to P. Furthermore, write A as the matrix of its row vectors

$$\begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{bmatrix}.$$

Then $PA = AM$ implies that

$$\begin{aligned}
a_2 &= a_1M,\\
a_3 &= a_2M,\\
&\cdots\cdots\\
a_n &= a_{n-1}M,\\
a_1 &= a_nM.
\end{aligned}$$

These in turn imply that

$$A = \begin{bmatrix} a_1 \\ a_1M \\ a_1M^2 \\ \cdots \\ a_1M^{n-1} \end{bmatrix}.$$

We now show that M is a generalized n-cycle. For, assume that M is the product of at least two disjoint cycles. We shall show that this implies det A is even, which is a contradiction since A is unimodular. In order to show this, it suffices to consider A modulo 2. We can then regard M as a permutation matrix. Number the elements so that the cycles are $(1\ 2 \cdots i_1)$ and $(i_1 + 1\ i_1 + 2 \cdots i_2)$. The first of the cycles affects only the first i_1 columns of A, the elements in each row of this submatrix being a permutation of $a_{11}, a_{12}, \cdots, a_{1i_1}$. The determinant of A therefore is divisible by $\sigma_1 = a_{11} + a_{12} + \cdots + a_{1i_1}$, modulo 2. Applying the same argument to the second cycle shows that det A is divisible by $\sigma_2 = a_{1i_1+1} + a_{1i_1+2} + \cdots + a_{1i_2}$, modulo 2, and applying the argument to the product of the cycles, we find that det A is also divisible by $\sigma_1 + \sigma_2$, modulo 2. At least one of the numbers $\sigma_1, \sigma_2, \sigma_1 + \sigma_2$ is even. This implies that A is singular modulo 2, so that M can not be the product of more than one disjoint cycle.

Since M and P are of equal determinant, Lemma 2 tells us that there is a generalized permutation matrix Q such that $M = Q'PQ$. Hence

$$A = \begin{bmatrix} a_1Q'Q \\ a_1Q'PQ \\ \cdots\cdots\cdots \\ a_1Q'P^{n-1}Q \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1P \\ \cdots\cdots \\ c_1P^{n-1} \end{bmatrix} \cdot Q = CQ,$$

with $c_1 = a_1Q'$.

This concludes the proof of Theorem 4′.

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Received April, 1955.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. IX, 93–113 (1956)

Singular Perturbations of Boundary Value Problems for Nonlinear Differential Equations of the Second Order*

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1. Introduction

This article is concerned with the asymptotic calculation of the solutions of boundary problems for nonlinear differential equations of the second order depending in a singular manner on a small parameter ε. More precisely, we shall study real differential equations of the type

$$\varepsilon y'' = F(x, y, y', \varepsilon) \tag{1.1}$$

together with boundary conditions of the form

$$y(\alpha) = l_1, \qquad y(\beta) = l_2, \tag{1.2}$$

where α, β, l_1, l_2 are constants independent of ε. The precise assumptions on $F(x, y, y', \varepsilon)$ will be found below. The most restrictive one is that $F(x, y, y', \varepsilon)$ *must be linear in* y', i.e. that

$$F(x, y, y', \varepsilon) = F_1(x, y, \varepsilon)y' + F_2(x, y, \varepsilon). \tag{1.3}$$

If the "reduced" differential equation $F(x, y, y', 0) = 0$ possesses a solution $y_0(x)$ satisfying one of the two boundary conditions (1.2) it is natural to ask whether the full problem has a solution $y(x, \varepsilon)$ that tends to $y_0(x)$ as $\varepsilon \to 0$ and to attempt a construction of $y(x, \varepsilon)$ by means of some perturbation scheme.

The questions concerning the existence, uniqueness and convergence, as $\varepsilon \to 0$, of solutions of (1.1) and (1.2) have been dealt with by E. Coddington and N. Levinson [1] and, in an earlier but less complete paper, by R. von Mises [2]. N. I. Brish [3] using a result of M. Nagumo [4] has proved somewhat more general results when F is of the form $F(x, y, y', \varepsilon) = F_1(x, y)y' + F_2(x, y, y')$ with $F_2(x, y, y')$ bounded for all y'. As pointed out

*The preparation of this paper was sponsored in part by the Office of Naval Research, United States Navy.

in [1] the boundary value problem cannot be expected to possess a solution when F is more strongly nonlinear in y'.

The present paper differs from these investigations in that *it contains a constructional scheme for the representation of the solution by means of convergent and asymptotic series which reveal its analytic structure.* The calculation of the terms of these series can be effected by quadratures. No use is made of the results of [1], so that our results furnish as a side product a new existence proof. The construction of our series requires analyticity of $F(x, y, y', \varepsilon)$ with respect to y, which is more restrictive than the hypotheses of [1] and [3], as far as y is concerned. On the other hand, it is assumed in [1] and [3] that F is independent of ε. It is not difficult to extend our proof to the case that α, β, l_1, l_2 are analytic functions of ε. Generalizations to certain higher order differential equations are very likely possible.

As in most perturbation problems in which a small parameter multiplies the highest derivative, there occurs, near one endpoint of the interval a phenomenon of non-uniform convergence, as $\varepsilon \to 0$, which is frequently called a boundary layer phenomenon, because the boundary layers of viscous flows are the best known physical applications of such mathematical occurrences. Usually, two analytic representations are needed in such problems, one valid in the boundary layer, the other one outside, and to calculate the correct transition from one region to the other is then an involved and difficult task. For *linear* ordinary differential equations asymptotic expressions valid umformly in the whole closed interval under consideration have been calculated before [5]. But the author is not aware of such representations for nonlinear equations. There exists, however, a literature on the asymptotic representation of solutions of *initial* value problems for nonlinear differential equations without a parameter, which uses methods related to those of the present paper. See, for instance, the articles [6], [7], [8] by Trjitzinsky and by Malmquist. There are also points of contact with the investigations of I. M. Volk [9].

2. Statement of the Problem

To fix the ideas let us assume that $\varepsilon > 0$. The modifications for $\varepsilon < 0$ are obvious. Our perturbation method is then based on the following assumptions.

ASSUMPTION A. *The "reduced differential equation"*

$$F_1(x, y, 0)y' + F_2(x, y, 0) = 0 \tag{2.1}$$

possesses a solution

$$y = y_0(x) \tag{2.2}$$

for which

$$y_0(\beta) = l_2 \text{ and } F_1(x, y_0(x), 0) < 0, \qquad \alpha \leqq x \leqq \beta. \tag{2.3}$$

If the solution $y_0(x)$ satisfies the conditions

$$y_0(\alpha) = l_1 \text{ and } F_1(x, y_0(x), 0) > 0, \qquad \alpha \leqq x \leqq \beta,$$

instead of (2.3), the transformation $x^* = -x$, $y^* = -y$ changes the given problem into one for which Assumption A is true.

ASSUMPTION B. *The functions $F_j(x, y, \varepsilon)$, $j = 1, 2$, are regular analytic with respect to y and ε and of class $C^{(2)}$ with respect to x in a region R of the x, y, ε-space that contains in its interior all points $y = y_0(x)$, $\alpha \leqq x \leqq \beta$, $\varepsilon = 0$.*

The assumption of analyticity with respect to ε can be replaced, without additional complications by the milder one that the $F_j(x, y, \varepsilon)$ admit asymptotic expansions in powers of ε.

Without loss of generality we may assume

$$\alpha = 0, \qquad \beta = 1, \tag{2.4}$$

$$y_0(x) \equiv 0, \tag{2.5}$$

and

$$F(x, 0, y', 0) = -y'. \tag{2.6}$$

For, the transformation of the variables x, y, ε into x^*, y^*, ε^* by means of the equations

$$x^* = \int_\alpha^x F_1(t, y_0(t), 0)\, dt \Big/ \int_\alpha^\beta F_1(t, y_0(t), 0)\, dt, \tag{2.7}$$

$$y^* = y - y_0(x), \tag{2.8}$$

$$\varepsilon^* = -\varepsilon \Big/ \int_\alpha^\beta F_1(t, y_0(t), 0)\, dt, \tag{2.9}$$

reduces the given problem to one having these properties, as can be seen by direct verification.

With these assumptions the differential equation (1.1) can be written in the form

$$\varepsilon y'' + p(x, \varepsilon) y' + q(x, \varepsilon) y = \varepsilon a(x, \varepsilon) + f(x, y, \varepsilon) y y' + g(x, y, \varepsilon) y^2, \tag{2.10}$$

where all coefficients in R are regular analytic with respect to y and ε and in class $C^{(2)}$ with respect to x. Moreover,

$$p(x, 0) = 1. \tag{2.11}$$

Our aim is to construct a solution of the differential equation (2.10) that

assumes the prescribed boundary values

$$y(0, \varepsilon) = y^0, \qquad y(1, \varepsilon) = 0 \tag{2.12}$$

and tends to zero in $0 < x \leqq 1$, as $\varepsilon \to 0$.

3. A Preliminary Transformation

Our construction of the required solution consists of two parts. In the first part we calculate a particular solution $y_1(x, \varepsilon)$ of the differential equation (2.10) that passes through the right endpoint $x = 1$, $y = 0$, but not generally through the left endpoint $x = 0$, $y = y^0$, and which has the property of being in the whole closed interval $0 \leqq x \leqq 1$ asymptotically representable by an ordinary power series in ε. This solution has no "boundary layer" for small ε at $x = 0$. In the second part it is shown that the required solution, which passes also through $x = 0$, $y = y^0$, differs from $y(x, \varepsilon)$ by a convergent series that is asymptotically zero except in an infinitesimal interval near $x = 0$.

In the present section the differential equation will be subjected to a preparatory transformation consisting in the subtraction of a certain function $y_1^*(x, \varepsilon)$ which will later be shown to be asymptotically equal to $y_1(x, \varepsilon)$.

We start from the observation that (2.10) can be *formally* satisfied by a series of the form

$$\sum_{r=1}^{\infty} v_r(x)\, \varepsilon^r, \tag{3.1}$$

since insertion of this series into (2.10) followed by expansion and identification of like powers of ε leads to recursion formulas of the form

$$v_1' + q(x, 0)v_1 = a(x, 0), \tag{3.2}$$

$$v_r' + q(x, 0)v_r = -v_{r-1}'' + k_r(x, v_1, \cdots, v_{r-1}, v_1', \cdots, v_{r-1}'), \quad r > 1, \tag{3.3}$$

which can be solved successively. We can—and will—even impose the boundary condition

$$v_r(1) = 0, \qquad r = 1, 2, \cdots. \tag{3.4}$$

The series so determined is in general divergent but by virtue of a theorem, a proof of which can e.g. be found in [10], there exists a function $y_1^*(x, \varepsilon)$—in fact, infinitely many such functions—analytic in ε and of class $C^{(2)}$ with respect to x, in $0 \leqq x \leqq 1$, $0 < \varepsilon \leqq \varepsilon_1$ (ε_1 a constant), that satisfies the asymptotic relation

$$y_1^*(x, \varepsilon) \sim \sum_{r=1}^{\infty} v_r(x)\, \varepsilon^r, \qquad 0 \leqq x \leqq 1, \quad 0 \leqq \varepsilon \leqq \varepsilon_1. \tag{3.5}$$

Also, this asymptotic relation may be twice term-wise differentiated with respect to x. (The proof in [10] deals only with asymptotic series with constant coefficients. But the additional facts concerning the differentiable dependence on x require only a simple corollary.) Because of (3.4) and the construction in [10] we have also

$$y_1^*(1, \varepsilon) = 0. \tag{3.6}$$

The function $y_1^*(x, \varepsilon)$ is in general not a solution of the differential equation (2.10), but with its help we shall be able to show the existence of a true solution $y_1(x, \varepsilon)$ for which

$$y_1(x, \varepsilon) \sim y_1^*(x, \varepsilon), \quad 0 \leqq x \leqq 1, \quad 0 \leqq \varepsilon \leqq \varepsilon_1, \tag{3.7}$$

and

$$y_1(1, \varepsilon) = 0. \tag{3.8}$$

In order to achieve this we introduce

$$z = y - y_1^*(x, \varepsilon) \tag{3.9}$$

as new dependent variable into (2.10) and show that the resulting differential equation for z possesses a solution with

$$z(x, \varepsilon) \sim 0, \qquad 0 \leqq x \leqq 1, \quad 0 \leqq \varepsilon \leqq \varepsilon_1, \tag{3.10}$$

$$z(1, \varepsilon) = 0. \tag{3.11}$$

The transformation results in a differential equation of the form

$$\varepsilon z'' + p(x, \varepsilon)z' + q(x, \varepsilon)z = \alpha(x, \varepsilon) + h(x, z, z', \varepsilon) \tag{3.12}$$

where

$$\begin{aligned} \alpha(x, \varepsilon) = f(x, y_1^*, \varepsilon)\, y_1^* y_1^{*\prime} + g(x, y_1^*, \varepsilon)\, y_1^{*2} + \varepsilon a(x, \varepsilon) \\ - \{\varepsilon y_1^{*\prime\prime} + p(x, \varepsilon) y_1^{*\prime} + q(x, \varepsilon) y_1^*\} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} h(x, z, z', \varepsilon) = f(x, y_1^*+z, \varepsilon)(y_1^*+z)(y_1^{*\prime}+z') + g(x, y_1^*+z, \varepsilon)(y_1^*+z)^2 \\ - f(x, y_1^*, \varepsilon)\, y_1^* y_1^{*\prime} - g(x, y_1^*, \varepsilon)\, y_1^{*2}. \end{aligned} \tag{3.14}$$

Since $y_1^*(x, \varepsilon)$ has the asymptotic expansion $\sum_{r=1}^{\infty} v_r(x)\varepsilon^r$ which satisfies (2.10) formally, it follows that

$$\alpha(x, \varepsilon) \sim 0, \qquad 0 \leqq \varepsilon \leqq \varepsilon_1. \tag{3.15}$$

Expanding the right member of (3.14) with respect to powers of z and

considering that ε can be factored from the right hand side of (3.5), we see that

$$h(x, z, z', \varepsilon) = \varepsilon(\pi(x, \varepsilon)z' + \varrho(x, \varepsilon)z) + h^*(x, z, z', \varepsilon) \tag{3.16}$$

with

$$h^*(x, z, z', \varepsilon) = f^*(x, z, \varepsilon)zz' + g^*(x, z, \varepsilon)z^2. \tag{3.17}$$

Here π, ϱ, f^* and g^* have asymptotic power series expansions in ε and are in class $C^{(2)}$ with respect to x. Moreover, f^* and g^* are analytic in z at $z = 0$. Combination of (3.16) and (3.17) with (3.12) leads to the form

$$\varepsilon z'' + p^*(x, \varepsilon)z' + q^*(x, \varepsilon)z = \alpha(x, \varepsilon) + h^*(x, z, z', \varepsilon) \tag{3.18}$$

for our differential equation, which is the basis of our further analysis. We observe that

$$p^*(x, 0) = p(x, 0) = 1, \qquad q^*(x, 0) = q(x, 0), \tag{3.19}$$

because of the factor ε in the right member of (3.16).

Our subsequent arguments are based on the asymptotic properties of *linear* differential equations whose homogeneous part is the left member of (3.18). We therefore collect in the next section some lemmas concerning such equations.

4. Lemmas on Linear Equations

In the following the symbol $[f(x)]$ will be used to designate functions of the form

$$[f(x)] = f(x) + \varepsilon E(x, \varepsilon),$$

where $E(x, \varepsilon)$ and its first derivative with respect to x are bounded in $0 \leqq x \leqq 1$, $0 \leqq \varepsilon \leqq \varepsilon_1$.

The first lemma below is a direct application of the standard asymptotic theory for linear differential equations (see [11] and [5]).

LEMMA 4.4. *The "variational differential equation"*

$$\varepsilon V'' + p^*(x, \varepsilon)V' + q^*(x, \varepsilon)V = 0 \tag{4.1}$$

possesses two solutions of the form

$$V_1(x, \varepsilon) = e^{-\sigma x}[v^*(x)], \qquad V_2(x, \varepsilon) = [v(x)] \tag{4.2}$$

where

$$\sigma = \varepsilon^{-1}, \tag{4.3}$$

$$v(x) = \exp\left\{\int_0^x q^*(t, 0)\, dt\right\}, \tag{4.4}$$

and

$$v^*(x) = v^{-1}(x) \exp\left\{-\int_0^x \pi(t, 0)dt\right\}.$$

COROLLARY. *In terms of its boundary values* $V(0, \varepsilon)$, $V(1, \varepsilon)$ *every solution* $V(x, \varepsilon)$ *of* (4.1) *can be written in the form*

$$V(x, \varepsilon) = A_1(\varepsilon)V_1(x, \varepsilon) + A_2(\varepsilon)V_2(x, \varepsilon),$$

where

$$A_1(\varepsilon) = V(0, \varepsilon)[1] - V(1, \varepsilon)[v(1)^{-1}],$$
$$A_2(\varepsilon) = - V(0, \varepsilon)e^{-\sigma}[v^*(1)/v(1)] + V(1, \varepsilon)[v(1)^{-1}].$$

LEMMA 4.2. *The solution* $W_{x_0}(x, \varepsilon)$ *of the differential equation*

$$\varepsilon W'' + p^*(x, \varepsilon)W' + q^*(x, \varepsilon)W = \varphi(x, \varepsilon) \tag{4.5}$$

with

$$W_{x_0}(x_0, \varepsilon) = W'_{x_0}(x_0, \varepsilon) = 0 \tag{4.6}$$

is of the form

$$W_{x_0}(x, \varepsilon) = \int_{x_0}^{x} K(x, t, \varepsilon)\varphi(t, \varepsilon)\, dt \tag{4.7}$$

with

$$K(x, t, \varepsilon) = \left[\frac{v(x)}{v(t)}\right] - e^{-\sigma(x-t)}\left[\frac{v^*(x)}{v^*(t)}\right]. \tag{4.8}$$

Proof: An application of the method of variation of parameters shows that

$$W_{x_0}(x, \varepsilon) = \sigma \int_{x_0}^{x} \frac{V_1(t, \varepsilon)V_2(x, \varepsilon) - V_1(x, \varepsilon)V_2(t, \varepsilon)}{V_1(t, \varepsilon)V_2'(t, \varepsilon) - V_1'(t, \varepsilon)V_2(t, \varepsilon)}\varphi(t, \varepsilon)\, dt.$$

From this formula the statement of the lemma is obtained by substitution of (4.2) followed by some elementary manipulation.

LEMMA 4.3. *The solution of the differential equation*

$$\varepsilon W'' + p^*(x, \varepsilon)W' + q^*(x, \varepsilon)W = e^{-2\sigma x}\varphi(x, \varepsilon) \tag{4.9}$$

for which

$$W(0, \varepsilon) = W(1, \varepsilon) = 0 \tag{4.10}$$

is of the form

$$W(x, \varepsilon) = \varepsilon e^{-\sigma x}\omega(x, \varepsilon) \tag{4.11}$$

with

$$|\omega^{(\nu)}(x, \varepsilon)| \leqq \sigma^{\nu} c \sup_{\substack{0 \leqq x \leqq 1 \\ 0 \leqq \varepsilon \leqq \varepsilon_1}} |\varphi(x, \varepsilon)|, \qquad \nu = 0, 1. \tag{4.12}$$

The constant c *is independent of* $\varphi(x, \varepsilon)$ *and* ε.

Proof: The solution $V(x, \varepsilon)$ of (4.1) that assumes the boundary values

$$V(0, \varepsilon) = 1, \qquad V(1, \varepsilon) = 0 \tag{4.13}$$

is of the form

$$V(x, \varepsilon) = e^{-\sigma x}\{[v^*(x)] - e^{\sigma(x-1)}[v^*(1)v(x)/v(1)]\}. \tag{4.14}$$

This follows directly from the corollary to Lemma 4.1. Now,

$$W(x, \varepsilon) = W_1(x, \varepsilon) - W_1(0, \varepsilon)V(x, \varepsilon), \tag{4.15}$$

and, by Lemma 4.2,

$$W_1(x, \varepsilon) = e^{-\sigma x}\int_1^x \left\{\left[\frac{v(x)}{v(t)}\right] e^{\sigma(x-2t)} - e^{-\sigma t}\left[\frac{v^*(x)}{v^*(t)}\right]\right\}\varphi(t, \varepsilon)\, dt. \tag{4.16}$$

Lemma 4.3 is a simple consequence of the last three formulas.

5. A Particular Solution without Boundary Layer

From Lemmas 4.1 and 4.2 it follows that, if the non-linear integral equation

$$\begin{aligned} z(x, \varepsilon) = {} & \int_0^x K(x, t, \varepsilon)\,\{\alpha(t, \varepsilon) + h^*(t, z(t, \varepsilon), z'(t, \varepsilon), \varepsilon)\}\,dt \\ & - \int_0^1 K(1, t, \varepsilon)\,\{\alpha(t, \varepsilon) + h^*(t, z(t, \varepsilon), z'(t, \varepsilon), \varepsilon)\}\,dt \cdot V_2^{-1}(1, \varepsilon)\, V_2(x, \varepsilon) \end{aligned} \tag{5.1}$$

possesses a solution, then it is also a solution of the differential equation (3.18). Moreover,

$$z(1, \varepsilon) = 0. \tag{5.2}$$

We shall now construct such a solution by Picard's iteration method and show that it is asymptotically equal to zero.

Let

$$z_0(x, \varepsilon) = 0, \tag{5.3}$$

and define $z_{n+1}(x, \varepsilon)$, $n = 0, 1, \cdots$, as the value of the right member of (5.1) when z and z' are replaced by z_n and z_n', respectively. Because of (3.15) and (3.17) we have from (5.1) that

$$z_1^{(\nu)}(x, \varepsilon) \sim 0, \qquad \nu = 0, 1, \quad 0 \leqq x \leqq 1, \quad 0 \leqq \varepsilon \leqq \varepsilon_1. \tag{5.4}$$

Generally,

$$
(5.5)\quad
\begin{aligned}
&z_{r+1}^{(\nu)}(x,\varepsilon) - z_r^{(\nu)}(x,\varepsilon)\\
&= \int_0^x K^{(\nu)}(x,t,\varepsilon)\{h^*(t, z_r(t,\varepsilon), z_r'(t,\varepsilon), \varepsilon) - h^*(t, z_{r-1}(t,\varepsilon), z_{r-1}'(t,\varepsilon), \varepsilon)\}\,dt\\
&\quad - \int_0^x K(1,t,\varepsilon)\,\{h^*(t, z_r(t,\varepsilon), z_r'(t,\varepsilon), \varepsilon)\\
&\quad - h^*(t, z_{r-1}(t,\varepsilon), z_{r-1}'(t,\varepsilon), \varepsilon)\}\,dt \cdot V_2^{-1}(1,\varepsilon)\,V_2^{(\nu)}(x,\varepsilon), \qquad \nu = 0, 1.
\end{aligned}
$$

The inequalities

$$
(5.6)\qquad |z_r^{(\nu)}(x,\varepsilon)| \leqq M\varepsilon^N, \quad \nu = 0, 1, \ 0 \leqq x \leqq 1, \ 0 < \varepsilon \leqq \varepsilon_1
$$

where N is an arbitrary positive integer and M a constant depending on N and ε_1, are certainly true for $r = 1$, in view of (5.4). Let us assume, for a proof by induction, that they have been proved for $r \leqq n$. It follows then from (3.17) that—at least for ε_1 so small that $z_r(x,\varepsilon)$ and $z_r'(x,\varepsilon)$ lie in the domain of regularity of $h^*(t, z, z', \varepsilon)$ —

$$
(5.7)\quad
\begin{aligned}
&|h^*(t, z_r(t,\varepsilon), z_r'(t,\varepsilon), \varepsilon) - h^*(t, z_{r-1}(t,\varepsilon), z_{r-1}'(t,\varepsilon), \varepsilon)|\\
&\leqq M_1\varepsilon(|z_r(t,\varepsilon) - z_{r-1}(t,\varepsilon)| + |z_r'(t,\varepsilon) - z_{r-1}'(t,\varepsilon)|),\\
&\qquad r \leqq n, \quad 0 \leqq t \leqq 1, \quad 0 \leqq \varepsilon \leqq \varepsilon_1.
\end{aligned}
$$

(The factor ε might even be replaced by ε^N.) M_1 depends on M. For abbreviation we introduce the notation

$$
(5.8)\qquad \zeta_r = \sup_{\substack{0\leq x\leq 1\\ 0\leq\varepsilon\leq\varepsilon_1}} (|z_r(t,\varepsilon) - z_{r-1}(t,\varepsilon)| + |z_r'(t,\varepsilon) - z_{r-1}'(t,\varepsilon)|).
$$

Insertion of (5.7) and (4.8) into (5.5) followed by summation over ν leads to

$$
(5.9)\qquad \zeta_{r+1} \leqq M_2\varepsilon\zeta_r, \qquad r \leqq n, \quad 0 \leqq \varepsilon \leqq \varepsilon_1,
$$

with another constant M_2, and hence to

$$
(5.10)\quad
\begin{aligned}
|z_{n+1}^{(\nu)}(x,\varepsilon)| &= \left|\sum_{r=1}^{n+1} z_r^{(\nu)}(x,\varepsilon) - z_{r-1}^{(\nu)}(x,\varepsilon)\right| \leqq \sum_{r=1}^{n+1} \zeta_r \leqq \sum_{r=1}^{n+1} (M_2\varepsilon)^r \zeta_1\\
&\leqq \frac{\varepsilon M_2\zeta_1}{1 - M_2\varepsilon}, \qquad \nu = 0, 1, \quad 0 \leqq x \leqq 1, \quad 0 \leqq \varepsilon \leqq \varepsilon_1.
\end{aligned}
$$

But

$$
\zeta_1 \leqq 4M\varepsilon^N,
$$

by virtue of (5.3) and (5.5); and (5.10) becomes, therefore,

$$
|z_{n+1}^{(\nu)}(x,\varepsilon)| \leqq \frac{4\varepsilon M_2}{1 - M_2\varepsilon} M\varepsilon^N, \qquad \nu = 0, 1, \ 0 \leqq x \leqq 1, \ 0 \leqq \varepsilon \leqq \varepsilon_1.
$$

The constants M, M_2 depend on ε_1, but they are bounded if ε_1 is small. For ε_1 sufficiently small, therefore, $4\varepsilon M_2/(1 - M_2\varepsilon)$ is less than unity, and (5.6) is proved for $r = n + 1$ also, i.e. it must be generally true.

For this value of ε_1 the factor $M_2\varepsilon$ in (5.9) is less than unity as well. From this fact the convergence of the sequence $z_n(x, \varepsilon)$ to a solution $z(x, \varepsilon)$ of the integral equation follows by the usual argument of Picard's iteration method. At the same time one proves, that $z_n'(x, \varepsilon)$ tends to $z'(x, \varepsilon)$. Finally, letting r tend to infinity in (5.6) we see that, since N is arbitrarily large,

$$z^{(\nu)}(x, \varepsilon) \sim 0, \qquad \nu = 0, 1, \quad 0 \leqq x \leqq 1, \quad 0 \leqq \varepsilon \leqq 1. \tag{5.11}$$

The second derivative $z''(x, \varepsilon)$ is then also asymptotically zero, in consequence of the differential equation itself. By repeated differentiation of the differential equation the formula (5.11) can be extended to as many derivatives as exist and are continuous.

The theorem below summarizes these results.

THEOREM 1. *The differential equation*

$$\varepsilon y'' + p(x, \varepsilon)y' + q(x, \varepsilon)y = \varepsilon a(x, \varepsilon) + f(x, y, \varepsilon)yy' + g(x, y, \varepsilon)y^2,$$

with $p(x, 0) = 1$, possesses, for $0 \leqq \varepsilon \leqq \varepsilon_1$, solutions that admit in $0 \leqq x \leqq 1$ a uniformly valid asymptotic representation by a series in powers of ε, as $\varepsilon \to +0$. Among these solutions there is one $y_1(x, \varepsilon)$, with $y_1(1, \varepsilon) = 0$. The asymptotic series may be twice termwise differentiated with respect to x.

We shall see later that $y_1(x, \varepsilon)$ is not the only such solution, but it is clear that all solutions that are asymptotically represented by a power series in $0 \leqq x \leqq 1$ and vanish at $x = 1$ can at most differ by functions that are asymptotically zero, since the coefficients of the power series must satisfy the differential equations (3.2), (3.3).

6. Construction of the Boundary Layer Correction

In Section 3 the function $y_1^*(x, \varepsilon)$ could have been *any* function satisfying the asymptotic relation (3.5). We may now take as $y_1^*(x, \varepsilon)$ in the transformation (3.9) the particular solution $y_1(x, \varepsilon)$ constructed in the preceding section. The transformation

$$z = y - y_1(x, \varepsilon)$$

leads then, as before, to the differential equation (3.18). Since $y_1(x, \varepsilon)$ is a true solution of the differential equation (2.10) we see from (3.13) that we now have

$$\alpha(x, \varepsilon) \equiv 0 \tag{6.1}$$

instead of the weaker asymptotic relation (3.15). To the required solution of

(2.10) with boundary values $(0, y^0)$ and $(1, 0)$ there corresponds a solution of (3.18) with

$$z(0, \varepsilon) = y^0 - y_1(0, \varepsilon), \qquad z(1, \varepsilon) = 0. \tag{6.2}$$

The existence and analytic structure of $z(x, \varepsilon)$ could probably be investigated by Picard's method making use of Green's function. This method would have the advantage of being applicable even when the differential equation is not analytic in y. But since the necessary calculations seem to be rather involved, and since we are aiming at an analytic expression in series form we shall find the solution instead in the form of a power series in its initial value $z(0, \varepsilon)$ and prove the convergence by the method of dominating series.

Let us set, for abbreviation,

$$y^0 - y_1(0, \varepsilon) = \mu \tag{6.3}$$

and insert for z in (3.18) formally the series

$$z = \sum_{r=1}^{\infty} u_r \mu^r, \tag{6.4}$$

where $u_r = u_r(x, \varepsilon)$ are functions to be determined. By identification of like powers of μ and by using (6.1), we obtain the recursive differential equations

$$\varepsilon u_1'' + p^*(x, \varepsilon)u_1' + q^*(x, \varepsilon)u_1 = 0, \tag{6.5}$$

$$\varepsilon u_r'' + p^*(x, \varepsilon)u_r' + q^*(x, \varepsilon)u_r = h_r(x, u_1, \cdots, u_{r-1}, u_1', \cdots, u_{r-1}', \varepsilon), \quad r > 1, \tag{6.6}$$

where the functions h_r have the following properties:

1) h_r is a polynomial in the u_j and u_j', $j = 1, \cdots, r-1$, linear in the u_j' combined and without constant or linear terms in the u_j and u_j' combined.

2) The coefficients of this polynomial possess asymptotic power series in ε each term of which is a function of x of class $C^{(2)}$.

If the u_r are determined from these differential equations, and if the resulting series (6.4), as well as its termwise derivative with respect to x converge uniformly and absolutely in $0 \leqq x \leqq 1$, then it represents there a solution of the differential equation (3.18) with $\alpha(x, \varepsilon) \equiv 0$. In fact, if we multiply the equations (6.5) and (6.6) by μ and μ^r, respectively, and sum over r, the right member is, in view of (6.4), precisely the function $h^*(x, z(x, \varepsilon), z'(x, \varepsilon), \varepsilon)$, and the last two terms in the left member become $p^*(x, \varepsilon)z' + q^*(x, \varepsilon)z$. The first term of the left member, i.e. $\varepsilon \sum_{r=1}^{\infty} u_r'' u^r$, must therefore also be uniformly convergent and, hence, equals $\varepsilon z''$.

The boundary conditions (6.2) are satisfied if we require that

$$u_1(0, \varepsilon) = 1, \qquad u_1(1, \varepsilon) = 0, \tag{6.7}$$

$$u_r(0, \varepsilon) = u_r(1, \varepsilon) = 0, \qquad r > 1. \tag{6.8}$$

7. Convergence Proof

The asymptotic form of $u_1(x, \varepsilon)$ is immediately determined from (6.5) and (6.7) with the help of (4.13) and (4.14). We find

$$\begin{aligned} u_1(x, \varepsilon) &= e^{-\sigma x}\{[v^*(x)] - e^{\sigma(x-1)}[v^*(1)v(x)/v(1)]\} \\ &= e^{-\sigma x}\eta_1(x, \varepsilon), \qquad 0 \leqq \varepsilon \leqq \varepsilon_1 \end{aligned} \tag{7.1}$$

where

$$|\eta_1^{(\nu)}(x, \varepsilon)| \leqq \sigma^\nu \hat{u}_1, \qquad \nu = 0, 1, \tag{7.2}$$

$\hat{u}_1$ being a constant independent of ε.

The convergence of the series (6.4) will be proved by constructing a dominating series $\sum_{r=1}^{\infty} \hat{u}_r \mu^r$ with constant positive coefficients. As a first step we construct a function $\hat{h}(z, z')$ which dominates $h^*(x, z, z', \varepsilon)$. Let the constants $z_1 > 0$ and ε_1 be so small that the two series

$$f^*(x, z, \varepsilon) = \sum_{s=0}^{\infty} \gamma_s(x, \varepsilon) z^s, \qquad g^*(x, z, \varepsilon) = \sum_{s=0}^{\infty} \delta_s(x, \varepsilon) z^s \tag{7.3}$$

for the functions in (3.17) converge uniformly and absolutely for

$$0 \leqq x \leqq 1, \quad |z| \leqq z_1, \quad 0 \leqq \varepsilon \leqq \varepsilon_1. \tag{7.4}$$

If C is a constant such that

$$|f^*(x, z, \varepsilon)| \leqq C, \qquad |g^*(x, z, \varepsilon)| \leqq C, \tag{7.5}$$

in that domain, then both functions are dominated by the series

$$C \sum_{s=0}^{\infty} \left(\frac{z}{z_1}\right)^s$$

which represents the function

$$C\left(1 - \frac{z}{z_1}\right)^{-1}.$$

Hence

$$\hat{h}(z, z') = C\left(1 - \frac{z}{z_1}\right)^{-1}(zz' + z^2) \tag{7.6}$$

dominates $h^*(x, z, z', \varepsilon)$, in the sense that every term in the series for h^*

in powers of z and z' has a coefficient whose absolute value in $0 \leqq x \leqq 1$, $0 \leqq \varepsilon \leqq \varepsilon_1$ has the corresponding coefficient in the expansion of $\hat{h}$ as upper bound.

Next, functions $\hat{h}_r$ dominating the functions h_r introduced in (6.6) are defined as follows: Insert into $\hat{h}(z, z')$ the formal series $\sum_{r=1}^{\infty} \hat{u}_r \mu^r$, $\sum_{r=1}^{\infty} \hat{v}_r \mu^r$ for z and z', respectively, and rearrange according to powers of μ. The coefficient of μ^r, which will be denoted by $\hat{h}_r(\hat{u}_1, \cdots, \hat{u}_{r-1}; \hat{v}_1, \cdots, \hat{v}_{r-1})$ is a polynomial in the indicated arguments, linear in the $\hat{v}_j$ with positive coefficients that are upper bounds, in $0 \leqq x \leqq 1$, $0 \leqq \varepsilon \leqq \varepsilon_1$, for the moduli of the corresponding coefficients of the polynomial $h_r(x, z_1, \cdots, z_{r-1}, z_1', \cdots, z_{r-1}', \varepsilon)$.

We now proceed to define a sequence $\hat{u}_r$ that dominates u_r. It has already been proved—cf. (7.1), (7.2)—that

$$(7.7)\quad |u_1(x, \varepsilon)| \leqq e^{-\sigma x}\hat{u}_1, \quad |u_1'(x, \varepsilon)| \leqq \sigma e^{-\sigma x}\hat{u}_1, \qquad 0 \leqq x \leqq 1,\ 0 \leqq \varepsilon \leqq \varepsilon_1.$$

We shall now show generally that

$$(7.8)\quad |u_r(x, \varepsilon)| \leqq e^{-\sigma x}\hat{u}_r, \quad |u_r'(x, \varepsilon)| \leqq \sigma e^{-\sigma x}\hat{u}_r, \qquad 0 \leqq x \leqq 1,\ 0 \leqq \varepsilon \leqq \varepsilon_1$$

where the $\hat{u}_r$ are determined from the recursive relation

$$(7.9)\qquad \hat{u}_r = c\hat{h}_r(\hat{u}_1, \cdots, \hat{u}_{r-1}; \hat{u}_1, \cdots, \hat{u}_{r-1}), \qquad r = 2, 3, \cdots.$$

Consider first $h_2(x, u_1, u_1', \varepsilon)$. This function is a polynomial in u_1 and u_1' without constant or linear terms and linear in u_1'. Hence, we obtain from (7.7) and the definition of $\hat{h}_2(\hat{u}_1, \hat{v}_1)$ the inequality

$$|h_2(x, u_1, u_1', \varepsilon)| \leqq \sigma e^{-2\sigma x}\hat{h}_2(\hat{u}_1, \hat{u}_1)$$

and therefore, by virtue of Lemma 4.3 and equations (7.9),

$$(7.10)\qquad |u_2^{(\nu)}(x, \varepsilon)| \leqq c\sigma^\nu e^{-\sigma x}\hat{h}_2(\hat{u}_1, \hat{u}_1) = \sigma^\nu e^{-\sigma x}\hat{u}_2.$$

The completion of the inductive proof of the formulas (7.8) is straightforward and is therefore omitted.

If we can show the convergence of the series $\sum_{r=1}^{\infty} \hat{u}_r \mu^r$, the uniform and absolute convergence of the series (6.4) will follow. Now, the equations (7.9) are also obtained, if the series $\sum_{r=1}^{\infty} \hat{u}_r \mu^r$ is inserted for ξ into the equation

$$(7.11)\qquad \xi - c\hat{h}(\xi, \xi) - \mu\hat{u}_1 = 0$$

and the coefficients of μ^r are set equal to zero. The equation (7.11) is satisfied for $\xi = \mu = 0$, and the partial derivative of the left member with

respect to ξ is different from zero at $\xi = \mu = 0$. Therefore (7.11) defines ξ as a regular analxtic function of μ in a neighborhood of $\mu = 0$. The series $\sum_{r=1}^{\infty} \hat{u}_r \mu^r$ represents this function, hence it converges for sufficiently small $|\mu|$, say $|\mu| \leqq \mu_0$.

The derived series $\sum_{r=1}^{\infty} u_r'(x, \varepsilon)\mu^r$ is dominated by $\sum_{r=1}^{\infty} \hat{u}_r \sigma \mu^r$. The series $\sum_{r=1}^{\infty} \varepsilon u_r'(x, \varepsilon)\mu^r$ is therefore also uniformly convergent in the same interval. It then follows, from the differential equation (1.1) itself and from the formal construction of the $u_r(x, \varepsilon)$ that the series $\sum_{r=2}^{\infty} \varepsilon^2 u_r''(x, \varepsilon)\mu^r$ converges uniformly as well. (The factors ε and ε^2 in these series are necessary, if we wish our statements to hold for $x = 0$ in the closed interval $0 \leqq \varepsilon \leqq \varepsilon_1$. The value of $\varepsilon u_r'(x, \varepsilon)$ at $\varepsilon = 0$ is of course to be defined by $\lim_{\varepsilon \to 0} \varepsilon u_r'(x, \varepsilon)$.)

Because of the factor $e^{-\sigma x}$ in (7.8) the values of the series $\sum_{r=1}^{\infty} u_r^{(\nu)}(x, \varepsilon)\mu^r$, $\nu = 0, 1$, tend exponentially to zero, in every interval $0 < x_1 \leqq x \leqq 1$.

In the theorem below these results are summarized in terms of the original differential equation (1.1). For convenience the notation has been changed in a few points.

THEOREM 2. *If the differential equation*

$$\varepsilon y'' = F_1(x, y, \varepsilon)y' + F_2(x, y, \varepsilon), \qquad \varepsilon > 0$$

satisfies Assumptions A and B of Section 2, then there exist constants $\varepsilon_1 > 0$, $\mu_1 > 0$ *such that the differential equation possesses, for* $0 < \varepsilon \leqq \varepsilon_1$, $|l_1 - y_0(0)| \leqq \mu_1$, *in* $\alpha \leqq x \leqq \beta$ *a solution satisfying the boundary condition*

$$y(\alpha, \varepsilon) = l_1, \qquad y(\beta, \varepsilon) = l_2.$$

This solution is representable by a uniformly and absolutely convergent series of the form

$$y(x, \varepsilon) = \sum_{r=0}^{\infty} u_r(x, \varepsilon)\mu^r. \tag{7.12}$$

The convergence is uniform with respect to ε, μ *and* x *for* $0 \leqq \varepsilon \leqq \varepsilon_1$, $|l_1 - y_0(\alpha)| \leqq \mu_1$, $\alpha \leqq x \leqq \beta$.

The terms of this series have the following properties:

1) $u_0(x, 0)$ *is the given solution* $y_0(x)$ *of the reduced equation which satisfies the right hand boundary condition* $u_0(\beta, 0) = l_2$.

2) $u_0(x, \varepsilon)$ *has an asymptotic series expansion in powers of* ε. *This series*

is a formal solution of the differential equation and satisfies the right hand boundary condition.

3) $\mu = l_1 - u_0(\alpha, \varepsilon)$.

4) *The functions* $u_r(x, \varepsilon)$, $r > 0$ *are of the form*

$$u_r(x, \varepsilon) = \exp\left\{-\sigma\int_\alpha^x F_1(t, y_0(t), 0)\, dt\right\}\omega_r(x, \varepsilon) \tag{7.13}$$

where $\omega_r(x, \varepsilon)$ *is bounded.*

The series (7.12) *may be twice termwise differentiated with respect to* x, *for* $0 < \varepsilon \leqq \varepsilon_1$. *The series* $\sum \varepsilon^r u_r^{(\nu)}(x, \varepsilon)\mu^r$, $\nu = 0, 1, 2$ *converge uniformly in the full range* $0 \leqq \varepsilon \leqq \varepsilon_1$, $|l_1 - y_0(\alpha)| \leqq \mu_1$, $\alpha \leqq x \leqq \beta$.

REMARKS.

1) *The boundary layer.* When $\mu = 0$ then $y^{(\nu)}(x, \varepsilon)$ tends to $y_0^{(\nu)}(x)$ uniformly in the *whole* interval $0 \leqq x \leqq 1$, for $\nu = 0, 1, 2, \cdots, n$, where $n - 2$ is the order of differentiability of $F(x, y, y', \varepsilon)$ with respect to x. In other words, there is then no boundary layer phenomenon. This is a little surprising, for one often hears the remark that the boundary layer occurs, because the solutions of the reduced *first* order equation cannot satisfy *both* prescribed boundary conditions. From this view point one would expect the boundary layer to disappear when the solution $y_0(x) = u_0(x, 0)$ of the reduced equation happens to pass through the left boundary point as well, i.e. when $l_1 - u_0(\alpha, 0) = 0$. However, $\mu = 0$ means that $l_1 - u_0(\alpha, \varepsilon) = 0$ or $l_1 - u_0(\alpha, 0) = O(\varepsilon)$. Thus, when both boundary conditions are satisfied by $y_0(x)$, Theorem 2 shows that

$$y(x, \varepsilon) = y_0(x) + O(\varepsilon) + \exp\left\{-\sigma\int_\alpha^x F_1(t, y_0(t), 0)\, dt\right\}O(\varepsilon).$$

Therefore, there is no boundary layer in $y(x, \varepsilon)$ itself, but—unless, exceptionally, the $O(\varepsilon)$ in the third term is $O(\varepsilon^2)$—it is still true that

$$\lim_{\varepsilon\to 0} y'(\alpha, \varepsilon) \neq y_0'(\alpha).$$

2) *The leading perturbation terms.* The calculation of the series (7.12) requires the successive solution of sequences of linear differential equations provided $y_0(x)$ is given. The terms of the asymptotic series for $u_0(x, \varepsilon)$ can be determined by quadratures. In view of formula (7.1) the same is true of $u_1(x, 0)e^{\sigma x}$. We give the leading terms explicitly: Let

$$\begin{aligned} p_1(x) &= F_y(x, y_0(x), y_0'(x), 0), \\ p_2(x) &= F_1(x, y_0(x), 0), \\ a(x) &= F_\varepsilon(x, y_0(x), y_0'(x), 0), \\ v_1(x) &= \int_x^\beta \exp\left\{\int_x^t p_1 p_2^{-1}\, dt\right\} a\, dt. \end{aligned}$$

(The subscripts y, ε indicate partial differentiations.) Then the solution of (1.1) satisfying (1.2) is approximately equal to

$$y_0(x) + \varepsilon v_1(x) + (l_1 - y_0(\alpha) - \varepsilon v_1(\alpha)) \exp\left\{\int_\alpha^x (\sigma p_2 + p_1 p_2^{-1})\, dt\right\}.$$

We omit the straightforward calculation that leads to this formula. The error is $O(\varepsilon^2)$ in every fixed subinterval $0 < x_1 \leqq x \leqq 1$. In the whole interval $0 \leqq x \leqq 1$ the error is $O((l_1 - y_0(\alpha))^2 + \varepsilon^2)$.

3) *The higher perturbation terms.* If the asymptotic series for $u_0(x, \varepsilon)$ has been calculated to within terms of order $O(\varepsilon^N)$—which, by virtue of the formulas (3.2), (3.3) and (3.4), requires only quadratures, no matter how large N—then the coefficients of the differential equations (6.5), (6.6) can also be determined with the same accuracy. Solution of these linear differential equations yields the functions $\omega_r(x, \varepsilon)$ of (7.13) to within terms of the same order. If $F(x, y, y', \varepsilon)$ is analytic in x, then the functions $[v^*(x)]$ and $[v(x)]$ of (4.2) possess asymptotic power series in ε, [12], and the approximate solution of (6.5) and (6.6) to any desired accuracy can also be performed by quadratures.

8. Reordering of the Series

If $F(x, y, y', \varepsilon)$ is analytic in x also, our results can be made more explicit on the basis of the following lemma.

LEMMA 8.1. *Let $\varphi(z, \varepsilon)$ be a function for which, as $\varepsilon \to +0$ an asymptotic expansion of the form*

$$\varphi(z, \varepsilon) \sim \sum_{r=0}^{\infty} \varepsilon^r \varphi_r(z)$$

is uniformly valid and indefinitely termwise differentiable with respect to z, in $0 \leqq z \leqq 1$; then

$$\int_0^z e^{m\sigma(t-z)} \varphi(t, \varepsilon)\, dt = \varepsilon \psi(z, \varepsilon) - \varepsilon e^{-m\sigma z} \chi(\varepsilon), \tag{8.1}$$

where $\psi(z, \varepsilon)$ also possesses an asymptotic expansion, whose leading term is $\varphi_0(z)$. Moreover, $\chi(\varepsilon) \sim \psi(0, \varepsilon)$.

Proof: By means of $n + 1$ integrations by parts one finds that

$$\begin{aligned}\int_0^z & e^{m\sigma(t-z)} \varphi(t, \varepsilon) dt \\ &= \frac{\varepsilon}{m} \sum_{s=0}^{n} (-1)^s \left(\frac{\varepsilon}{m}\right)^s \varphi^{(s)}(z, \varepsilon) - \frac{\varepsilon}{m} e^{-m\sigma z} \sum_{s=0}^{n} (-1)^s \left(\frac{\varepsilon}{m}\right)^s \varphi^{(s)}(0, \varepsilon) \\ &\quad + (-1)^{n+1} \left(\frac{\varepsilon}{m}\right)^{n+1} \int_0^z e^{m\sigma(t-z)} \varphi^{(n+1)}(t, \varepsilon)\, dt.\end{aligned} \tag{8.2}$$

One more integration by parts shows that the last term of the right member is $O(\varepsilon^{n+2})$, uniformly in $0 \leqq z \leqq 1$. Now, by assumption,

$$\varphi^{(s)}(z, \varepsilon) = \sum_{r=0}^{n} \varepsilon^r \varphi_r^{(s)}(z) + \varepsilon^{n+1} E_n^{(s)}(z, \varepsilon), \tag{8.3}$$

where each $E_n^{(s)}(z, \varepsilon)$ is uniformly bounded in $0 \leqq \varepsilon \leqq \varepsilon_1$, $0 \leqq x \leqq 1$. Inserting (8.3) into (8.2) and rearranging according to powers of ε we obtain a relation of the form

$$\int_0^z e^{m\sigma(t-z)} \varphi(t, \varepsilon)\, dt = \varepsilon \sum_{\nu=0}^{n} \varepsilon^\nu \psi_\nu(z) - \varepsilon e^{-m\sigma z} \sum_{\nu=0}^{n} \varepsilon^\nu \psi_\nu(0) + O(\varepsilon^{n+1}), \tag{8.4}$$

with $\psi_0(z) = \varphi_0(z)$. Let $\psi(z, \varepsilon)$ denote some function for which

$$\psi(z, \varepsilon) \sim \sum_{\nu=0}^{\infty} \varepsilon^\nu \psi_\nu(z), \tag{8.5}$$

uniformly in $0 \leqq z \leqq 1$. The existence of such a function is assured by the theorem already used in formula (3.5). Since the sums in the right member of (8.4) differ from $\psi(z, \varepsilon)$ or $\psi(0, \varepsilon)$ by terms of the order $O(\varepsilon^{n+1})$ and n is arbitrary, the proof of the lemma is at hand.

With the help of this lemma and those of Section 4 the analytic form of the functions $u_r(x, \varepsilon)$ can be studied more explicitly. The reasoning consists in a rather simple elaboration of the previous discussion of these functions. We content ourselves with the statement of the resulting formula

$$u_r(x, \varepsilon) = \sum_{s=0}^{r} \varphi_{rs}(x, \varepsilon) e^{-s\sigma x}, \tag{8.6}$$

(provided $\int_\alpha^x F_1(t, y_0(t), 0)dt = x$) in which all $\varphi_{rs}(x, \varepsilon)$ possess asymptotic power series in ε. In particular it turns out that

$$\varphi_{r0}(x, \varepsilon) \sim 0, \qquad r > 0. \tag{8.7}$$

The formula (8.6) suggests a rearrangement of the series $\sum_{r=0}^{\infty} u_r(x, \varepsilon)\mu^r$ according to powers of $e^{-\sigma x}$. This is permissible, if the series $\sum_{r=0}^{\infty} u_r^*(x, \varepsilon)\mu^r$ with

$$u_r^*(x, \varepsilon) = \sum_{s=0}^{r} |\varphi_{rs}(x, \varepsilon)|\, e^{-s\sigma x}$$

converges. That this is so requires again only a somewhat detailed retracing of the convergence proof for $\sum_{r=0}^{\infty} u_r(x, \varepsilon)\mu^r$, which will be omitted.

The rearrangement leads to a series of the form

$$\sum_{m=0}^{\infty} a_m(x, \varepsilon, l_1) e^{-m\sigma x}.$$

Because of (8.7) we have

$$a_0(x, \varepsilon, l_1) \sim u_0(x, \varepsilon).$$

The functions $a_m(x, \varepsilon, l_1)$ are of the form

$$a_m(x, \varepsilon, l_1) = \sum_{s=m}^{\infty} (l_1 - u_0(\alpha, \varepsilon))^s \varrho_{sm}(x, \varepsilon)$$

with coefficients $\varrho_{sm}(x, \varepsilon)$ analytic in x and possessing asymptotic series in ε.

Analogous arguments can be applied to the series $\sum_{r=1}^{\infty} \varepsilon^r u_r^{(\nu)}(x, \varepsilon)\mu^r$, $(\nu = 1, 2)$. The theorem below states these results in the notation of the general case.

THEOREM 3. *If $F_1(x, y, \varepsilon)y + F_2(x, y, \varepsilon)$ is analytic in x, in $0 \leqq x \leqq 1$, in addition to satisfying Assumptions A and B, then the solution (7.12) of the boundary value problem formulated in Theorem 2 is of the form*

$$y(x, \varepsilon) = \sum_{m=0}^{\infty} c_m(x, l_1, \varepsilon) \exp\left\{m\sigma \int_\alpha^x F_1(t, y_0(t), 0)\, dt\right\}. \tag{8.8}$$

The series converges uniformly and absolutely for $0 \leqq x \leqq 1$, $0 \leqq \varepsilon \leqq \varepsilon_1$, $|l_1 - y_0(x)| < \mu_1$. The functions $c_m(x, l_1, \varepsilon)$ are there regular analytic in x and in l_1 and possess asymptotic series in powers of ε. Moreover

$$c_0(x, l_1, \varepsilon) \sim u_0(x, \varepsilon),$$

and

$$c_m(x, l_1, \varepsilon) = (l_1 - u_0(\alpha, \varepsilon))^m c_m^*(x, l_1, \varepsilon),$$

where $c_m^*(x, l_1, \varepsilon)$ has the same regularity properties as $c_m(x, l_1, \varepsilon)$.

The series

$$\sum_{m=0}^{\infty} \varepsilon^\nu \frac{d^\nu}{dx^\nu}\left[c_m(x, l_1, \varepsilon) \exp\left\{m\sigma \int_\alpha^x F_1(t, y_0(t), 0)\, dt\right\}\right], \qquad \nu = 1, 2, \tag{8.9}$$

converge uniformly in the same range.

9. Stretching of the Boundary Layer

Although the series representations in Theorems 2 and 3 are valid in the whole interval $\alpha \leqq x \leqq \beta$, they have certain disadvantages for the approximate calculation of the solution inside the boundary layer. The approximate formula derived in Remark 2 of Section 7, for instance, is satisfactory only, if $l_1 - y_0(\alpha)$ is small, more precisely, if it is of the order $O(\varepsilon)$. In the hydrodynamic applications, where such a condition is usually not satisfied, approximate solutions in the boundary layer are in general obtained by a „stretch-

ing" of the independent variable. In our example a natural stretching transformation consists in setting

$$(9.1) \qquad x = \alpha + z\varepsilon, \qquad y(x, \varepsilon) = y(\alpha + z\varepsilon, \varepsilon) = w(z, \varepsilon).$$

This changes the given differential equation into

$$(9.2) \qquad \frac{d^2w}{dz^2} = F_1(\alpha, w, 0)\frac{dw}{dz} + \varepsilon G\left(z, w, \frac{dw}{dz}, \varepsilon\right).$$

If $F(x, y, y', \varepsilon)$ is analytic in all variables, then $G\left(z, w, \frac{dw}{dz}, \varepsilon\right)$ is also. The boundary conditions (1.2) are changed into

$$w(0, \varepsilon) = l_1, \qquad w((\beta - \alpha)\sigma, \varepsilon) = l_2.$$

It is plausible, but not obvious, that $w(z, \varepsilon)$ can be approximated, at least for bounded z intervals, by some solution $w_0(z)$ of the differential equation

$$(9.3) \qquad \frac{d^2w_0}{dz^2} = F_1(\alpha, w_0, 0)\frac{dw_0}{dz}.$$

Another, even less self-evident, plausibility consideration leads to the boundary conditions

$$(9.4) \qquad w_0(0) = l_1, \qquad w_0(\infty) = y_0(\alpha)$$

for this approximate solution.

We shall now confirm these conjectures with the help of Theorems 2 and 3. The series (8.8) for the full solution $y(x, \varepsilon)$ becomes, after the transformation (9.1),

$$(9.5) \quad w(z, \varepsilon) = \sum_{m=0}^{\infty} c_m(\alpha + \varepsilon z, l_1, \varepsilon) \exp\left\{m\int_0^z F_1(\alpha + \varepsilon\tau, y_0(\alpha + \varepsilon\tau), 0)\, d\tau\right\}.$$

According to Theorem 3 this series is convergent, uniformly with respect to z and to ε in the range

$$0 \leqq z \leqq (\beta - \alpha)\sigma, \qquad 0 \leqq \varepsilon \leqq \varepsilon_1.$$

For $\varepsilon = 0$ it defines, therefore, in particular, the function

$$(9.6) \qquad w(z, 0) = \sum_{m=0}^{\infty} c_m(\alpha, l_1, 0) \exp\{mz F_1(\alpha, y_0(\alpha), 0)\}.$$

Since

$$w(0, 0) = \sum_{m=0}^{\infty} c_m(\alpha, l_1, 0) = \lim_{\varepsilon \to 0} y(0, \varepsilon) = l_1,$$

$w(z, 0)$ has the initial value

$$(9.7) \qquad w(0, 0) = l_1.$$

As $z \to \infty$, we have, thanks to the condition (2.3),

$$w(\infty, 0) = c_0(\alpha, l_1, 0).$$

But by Theorem 3,

$$c_0(\alpha, l_1, 0) = u_0(\alpha, 0),$$

and by Theorem 2,

$$u_0(\alpha, 0) = y_0(\alpha).$$

Hence,

$$w(\infty, 0) = y_0(\alpha). \tag{9.8}$$

Thus $w(z, 0)$ satisfies the two boundary conditions for $w_0(z)$ in (9.4).

The series (8.9) becomes, after setting $x = \alpha + z\varepsilon$,

$$\sum_{m=0}^{\infty} \frac{d^\nu}{dz^\nu} c_m(\varepsilon z + \alpha, l_1, \varepsilon) \exp\left\{ m \int_0^z F_1(\alpha + \varepsilon\tau, y_0(\alpha + \varepsilon\tau), 0)\, d\tau \right\}, \qquad \nu = 1, 2.$$

This is the termwise derivative of the series (9.5); it is uniformly convergent, with respect to z, in any fixed interval $0 \leqq z \leqq z_1$. It therefore represents $d^\nu w(z, \varepsilon)/dz^\nu$. Since the convergence is uniform with respect to ε also, for $0 \leqq \varepsilon \leqq \varepsilon_1$, and as each term is continuous in ε at $\varepsilon = 0$, we have

$$\lim_{\varepsilon \to 0} \frac{d^\nu w(z, \varepsilon)}{dz^\nu} = \frac{d^\nu w(z, 0)}{dz^\nu}, \qquad \nu = 1, 2, \tag{9.9}$$

for $0 \leqq z \leqq z_1$.

Observe that the formally analogous relation

$$\lim_{\varepsilon \to 0} y^\nu(x, \varepsilon) = y^{(\nu)}(x, 0) \tag{9.10}$$

is *not* true at $x = \alpha$. The reason is, of course, that each term of the series (8.9) is discontinuous in ε and x combined at $\varepsilon = 0$, $x = \alpha$. In (9.10) the value of x is held fixed independently of ε, whereas in (9.9) the variable x approaches α in accordance with the relation $x = \alpha + \varepsilon z$. Thus we are dealing with two entirely different passages to the limit.

From (9.2) and (9.9) we conclude that $w(z, 0)$ satisfies the differential equation (9.3). We have therefore proved

THEOREM 4. *If $y(x, \varepsilon)$ is the solution* (7.12) *of the given differential equation with the boundary values* (1.2), *then*

$$\lim_{\varepsilon \to +0} y(\alpha + \varepsilon z, \varepsilon) = w_0(z),$$

uniformly for $0 \leqq z \leqq z_1$ (z_1 arbitrary but independent of ε), where $w_0(z)$ satisfies the differential equation (9.3) *and the boundary conditions* (9.4).

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Received April, 1955.